Microeconomics II

Spring 2025

Final Exam Solution Topics

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You have a total of 120 minutes (2 hours) to solve the exam.
Identify each sheet with your Student Number and Name.
Good luck!

I

See MWG 14.C.8 solutions

II (5 points)

The agent's utility is $u(t, l, k) = t + wl - kh(l)$.

So $U(\widehat{k}, k) = wl(\widehat{\theta}) - kh(l(\widehat{k})) + t(\widehat{k})$.

Let $U(k) \equiv U(k, k)$. Using the value function and incorporating the FOC (or just applying the envelope theorem), we have that

$$\frac{dU}{dk}(k) = \frac{\partial U}{\partial k}(k) = -h(l(k))$$

Note that $\frac{dU}{dk}(k) < 0$.

Since $\frac{\partial u}{\partial l} = w - k\frac{dh}{dl}$ and $\frac{\partial u}{\partial t} = 1$, we have that $\frac{d}{dk}(\frac{\frac{\partial u}{\partial l}}{\frac{\partial u}{\partial t}}) = -\frac{dh}{dl} < 0$ and CS^{-} holds. This condition that implies that any implementable contract must have the workload be a non-increasing function of type.

We have $\frac{dU}{dk}(k) < 0$ and therefore the "worst type" is type $\overline{k}$. Integrating from k to $\overline{k}$ both sides of the equation $\frac{dU}{dk}(k) = -h(l(k))$ yields $U(k) = U(\overline{k}) + \int_k^{\overline{k}} h(l(\widetilde{k}))d\widetilde{k}$. Therefore, we can rewrite $t(k) = U(\overline{k}) + \int_k^{\overline{k}} h(l(\widetilde{k}))d\widetilde{k} - wl(k) + kh(l(k))$

Given that $t(k)$ and therefore $U(\overline{k})$ enters the principal's objective function with a negative sign, the principal will want to set $t(\overline{k})$ so that $U(\overline{k}) = 0$ (IR is binding for the "worst type").

At the optimum, we have $U(k) = \int_k^{\overline{k}} h(l(\widetilde{k}))d\widetilde{k}$ and $t(k) = \int_k^{\overline{k}} h(l(\widetilde{k}))d\widetilde{k} - wl(k) + kh(l(k))$.

Replacing $t(k)$ in the firm's objective function (and ignoring the monotonicity constraint), the simplified problem is:

$$\max_{l(k)} E_k \left[wl(k) - kh(l(k)) - \int_k^{\overline{k}} h(l(\widetilde{k}))d\widetilde{k} \right]$$

Since

$$E_k \left[\int_k^{\overline{k}} h(l(\widetilde{k}))d\widetilde{k} \right] = \int_{\underline{k}}^{\overline{k}} \int_k^{\overline{k}} h(l(\widetilde{k}))d\widetilde{k} f(k)dk,$$

integration by parts yields

$$E_k \left[\int_k^{\overline{k}} h(l(\widetilde{k}))d\widetilde{k} \right] = E_k \left[h(l(k))\frac{F(k)}{f(k)} \right]$$

The principal's problem is then

$$\max_{l(k)} E_k \left[wl(k) - [k + \frac{F(k)}{f(k)}].h(l(k)) \right]$$

Pointwise differentiation yields:

$$w - [k + \frac{F(k)}{f(k)}]h'(l(k)) = 0$$

and

$$l(k) = h'^{-1} \left[\frac{w}{k + \frac{F(k)}{f(k)}} \right]$$

We can check that the monotonicity of $l(.)$ holds due to the assumption on the monotonicity of $\frac{F}{f}$ (and on the sign of h').Finally, the transfers can be calculated from $t(k) = \int_k^{\overline{k}} h(l(\widetilde{k}))d\widetilde{k} - wl(k) + kh(l(k))$.

b) At the first best, we get:

$$l(k) = h'^{-1} \left[\frac{w}{k} \right]$$

and for each k, the workload would be higher.

III (5.5 points)

a) Let θ denote the n-dimensional vector of buyer valuations where each θ_i follows a U[0,2] distribution. Let $T_i(\theta)$ denote the (possibly negative) transfer from the principal to agent i and let $X_i(\theta)$ denote the decision function (probability of the object going to agent i).

Let $x_i(\theta_i) = E_{\theta_{-i}}(X(\theta))$ and $t_i(\theta_i) = E_{\theta_{-i}}(T_i(\theta))$.

The utility of a buyer i is $u_i = t_i + \theta_i x_i$

(i) IC

For buyer i's IC, from $U_i(\widehat{\theta}_i, \theta_i) = t_i(\widehat{\theta}_i) + \theta_i x_i(\widehat{\theta}_i)$ we can write the value function $U_i(\theta_i) \equiv U_i(\theta_i, \theta_i)$. Using the value function and applying the envelope theorem (or just incorporating the FOC), we have that $\frac{dU_i}{d\theta_i}(\theta_i) = \frac{\partial u_i}{\partial \theta_i}(\theta_i)$ and

$$\frac{dU_i}{d\theta_i}(\theta_i) = x_i(\theta_i).$$

Since $\frac{d}{d\theta_i}\left(\frac{\frac{\partial u_i}{\partial x_i}}{\frac{\partial u_i}{\partial t_i}}\right) = 1 > 0$, CS^+ holds. Therefore, we need $x_i(\theta_i)$ to be nondecreasing in any implementable contract.

$$IC \iff \begin{cases} \frac{dU_i}{d\theta_i}(\theta_i) = x_i(\theta_i) \\ x_i(\theta_i) \text{ nondecreasing} \end{cases}$$

We can now rewrite the profits of buyer i as the sum of the profits of the "worst type" and an integral. Integrating $\frac{dU_i}{d\theta_i}(\theta_i) = x_i(\theta_i)$ from $\underline{\theta}_i = 0$ to θ_i yields $U_i(\theta_i) = U_i(0) + \int_0^{\theta_i} x_i(\widetilde{\theta}_i)d\widetilde{\theta}_i$. Since $U_i(\theta_i) = t_i(\theta_i) + \theta_i x_i(\theta_i)$, we can write $t_i(\theta_i) = U_i(0) + \int_0^{\theta_i} x_i(\widetilde{\theta}_i)d\widetilde{\theta}_i - \theta_i x_i(\theta_i)$.

(ii) IR just requires that $U_i(0) \geq 0$. Since the seller wants to minimize transfers, any optimal auction must lead to $U_i(0) = 0$.

The seller wants to minimize $\sum_i E_{\theta_i}[\int_0^{\theta_i} x_i(\widetilde{\theta}_i)d\widetilde{\theta}_i - \theta_i x_i(\theta_i)]$ (subject to the monotonicity constraints).

Since

$$E_{\theta_i}\left[\int_0^{\theta_i} x_i(\widetilde{\theta}_i)d\widetilde{\theta}_i\right] = \int_0^2 \int_0^{\theta_i} x_i(\widetilde{\theta}_i)d\widetilde{\theta}_i d\theta_i,$$

integration by parts yields

$$E_{\theta_i}\left[\int_0^{\theta_i} x_i(\widetilde{\theta}_i)d\widetilde{\theta}_i\right] = E_{\theta_i}\left[x_i(\theta_i).(2 - \theta_i)\right]$$

The simplified problem is then:

$$\min_{\{X_i(\theta)\}} \sum_i E_{\theta_i}[x_i(\theta_i)(2 - 2\theta_i] = \min_{\{X_i(\theta)\}} \sum_i E_\theta [X_i(\theta)(2 - 2\theta_i]$$

Let $J(\theta_i) \equiv 2 - 2\theta_i$.Then, $J(.)$ is decreasing. The optimal mechanism is the one that sets $X_i(\theta) = 1$ if $J(\theta_i) \leq 0$ and $J(\theta_i) \leq J(\theta_j)$ (and 0 otherwise). Since $J(1) = 0$, the optimal mechanism is: $X_i(\theta) = \begin{cases} 1 & \text{if } \theta_i = \max\{\theta_1, ..., \theta_n, 1\} \\ 0 & \text{otherwise} \end{cases}$

The optimal mechanism (that maximizes the seller's expected revenue) could be achieved with a second-price sealed-bid auction provided that there is a reservation price of 1. The value of the secret bid would then be 1. However, this auction would not be efficient, because the object might not be sold in case every agent has a values the object below 1.

b) Suppose that the website decides to switch to first-price sealed-bid auctions. Will her expected revenue be the same? Would your answer change if the n buyers were risk-averse?

Initially, the assumptions of the Revenue Equivalence Theorem all hold: risk-neutral agents, 0 utility for agents with 0 valuation and the same decision function. Therefore, expected revenue will be the same.

However, risk aversion will lead to different expected revenues under the two mechanisms: whereas for the second-price sealed-bid auction, the dominant strategy will still be to bid the true valuation, shading a bid in a first-price sealed-bid auction becomes more costly. Therefore, the expected revenue under the first-price sealed-bid auction would be greater.

IV (4.5 points)

a) Proof done in class.
b) Arrow's theorem applies to social welfare functions that satisfy Unrestricted Domain (Universal Domain), Pareto, Independence of Irrelevant Alternatives (IIA), Completeness and Transitivity – and non-dictatorship.
This functional applies to any profile of individual preferences; It satisfies Pareto (if everyone strictly prefers x to y, then no agent weakly preferes y to x and therefore society will also strictly prefer x to y) and IIA (only the individual rankings of x and y matter for the social ranking of x and y). The social preference ordering will be complete: if there is at least one agent weakly preferring x to y and at least one agent weakly preferring y to x, society will be indifferent between x and y; if all agents strictly prefer x to y, society will strictly prefer x to y. But the social preference ordering will not be transitive: suppose person 1 prefers x to y and person 2 prefers y to z, but everyone strictly prefers z to x; then xRyRz but zPx and transitivity is violated.