Problem Set 2
Solution topics

2.

a) If type is observable, I solve:

$$\max_{x_I, x_o} B_I(x_0) - c_I(x_I) \quad i = T, U$$

$$s.t.\ B_0(x_I) - c_0(x_0, \theta_i) \geq 0$$

Each type will get the net benefit of 0 (in order to satisfy the participation constraint).

$$\pounds = B_I(x_0) - c_I(x_I) + \lambda \left[B_0(x_I) - c_0(x_0, \theta_i) \right]$$

FOC

$$\frac{B_I'(x_0)}{c_0'(x_0, \theta_i)} = \frac{c_I'(x_I)}{B_0'(x_I)} \text{ or } \frac{B_I'(x_0)}{c_I'(x_I)} = \frac{c_0'(x_0, \theta_i)}{B_0'(x_I)}$$

(my marginal benefit, corrected by the price I have to pay, equals the other´s marginal cost, corrected by the price he receives)

b) If type is not observable, the first best contracts can not be offered - otherwise the true type would want to take the untrue type´s contract.

The problem is:

$$\max_{x_I^T, x_O^T, x_I^U, x_0^U} \lambda \left[B_I(x_0^T) - c_I(x_I^T) \right] + (1 - \lambda) \left[B_I(x_0^U) - c_I(x_I^U) \right]$$

$$\begin{array}{lrcl} IR_T & B_0(x_I^T) - c_0(x_0^T, \theta_T) & \geq & 0 \\ IR_U & B_0(x_I^U) - c_0(x_0^U, \theta_U) & \geq & 0 \\ IC_T & B_0(x_I^T) - c_0(x_0^T, \theta_T) & \geq & B_0(x_I^U) - c_0(x_0^U, \theta_T) \\ IC_U & B_0(x_I^U) - c_0(x_0^U, \theta_U) & \geq & B_0(x_I^T) - c_0(x_0^T, \theta_U) \end{array}$$

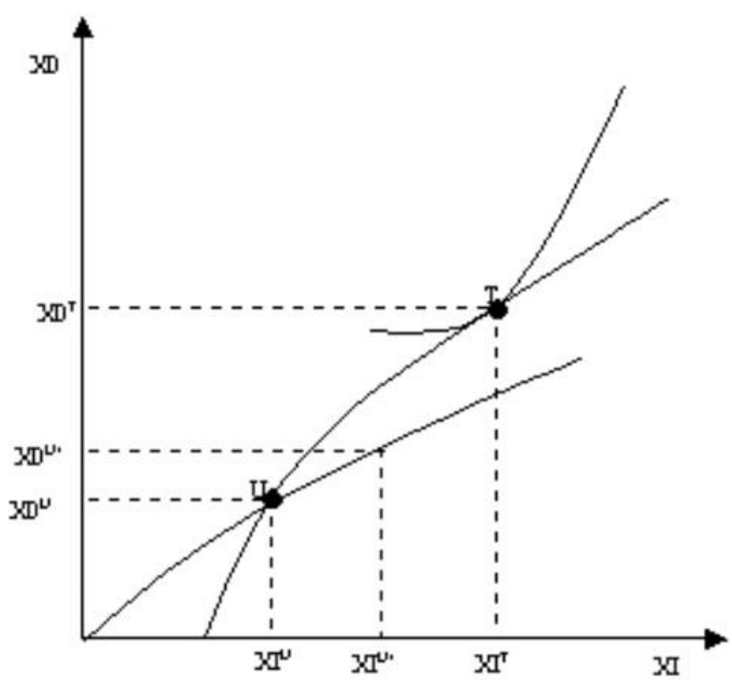
- IR_T is not binding at the optimum:
$B_0(x_I^T) - c_0(x_0^T, \theta_T) \geq B_0(x_I^U) - c_0(x_0^U, \theta_T) > B_0(x_I^U) - c_0(x_0^U, \theta_U) \geq 0$

-IR_U is binding at the optimum
Otherwise, I could decrease x_I^U and x_I^T so that $dB_0(x_I^T) = dB_0(x_I^U) = \varepsilon$ small enough so that IR constraints are still satisfied (and IC constraints do not change) and my net benefit would increase.

-IC_T is binding at the optimum
Otherwise, I could decrease x_I^T by an ε small enough that IR_T and IC_T are still met (clearly IC_U will be met as well and IR_U does not change) and increase my net benefit

- IC_U is not binding at the optimum
Using IR_U and IC_T in IC_U yields $0 \geq c_o\left(x_o^U, \theta_U\right) - c_o\left(x_o^T, \theta_U\right) - \left[c_o\left(x_o^U, \theta_T\right) - c_o\left(x_o^T, \theta_T\right)\right]$
The RHS is approximately:
$c_o`\,(., \theta_U)\left(x_o^U - x_o^T\right) - c_o'(., \theta_T)\left(x_o^U - x_o^T\right) = \left[c_o'(., \theta_U) - c_o'(., \theta_T)\right]\left(x_o^U - x_o^T\right) =$
$(+)(-)$

Graphically:
The optimal U contract must be on the reservation utility indifference curve and the optimal T contract must be on the indifference curve for type θ_T that goes through the U contract. Although $x_I^U < x_I^{U*}$ and $x_o^U < x_o^{U*}$ (if $x_I^U = x_I^{U*}$ and $x_o^U = x_o^{U*}$, I could decrease them slightly getting a second order decrease in my net benefit but getting a first order increase in the net benefit I derive from type θ_T), we are not sure whether $x_I^T = x_I^{T*}$ because the tangency between my indifference curve and type θ_T´s indifference curve is not always at the same level of x_I (or x_o, for that matters). Also, clearly, IC_U is not binding at the optimum.



Analytically, we have:

$$\max_{x_I^T, x_O^T, x_I^U, x_0^U} \lambda \left[B_I(x_0^T) - c_I(x_I^T)\right] + (1-\lambda)\left[B_I(x_0^U) - c_I(x_I^U)\right]$$

$$\begin{array}{lrcl} \alpha & B_0(x_I^U) - c_0(x_0^U, \theta_U) & = & 0 \\ \gamma & B_0(x_I^T) - c_0(x_0^T, \theta_T) & = & B_0(x_I^U) - c_0(x_0^U, \theta_T) \end{array}$$

FOC:

$$\begin{array}{lrcl} 1) & \lambda B_I'(x_0^T) - \gamma c_0'(x_0^T, \theta_T) & = & 0 \\ 2) & -\lambda c_I'(x_I^T) + \gamma B_0'(x_I^T) & = & 0 \\ 3) & (1-\lambda) B_I'(x_0^U) + \gamma c_0'(x_0^U, \theta_T) - \alpha c_0'(x_0^U, \theta_U) & = & 0 \\ 4) & -(1-\lambda) c_I'(x_I^U) - \gamma B_0'(x_I^U) + \alpha B_0'(x_I^U) & = & 0 \end{array}$$

1)+2) $\Rightarrow \frac{B_I'(x_0^T)}{c_0'(x_0^T, \theta_T)} = \frac{c_I'(x_I^T)}{B_0'(x_I^T)} \rightarrow$ same condition as in the observable case which is why we have efficiency at the top

We make the terms of the U contract worse and give the true type an informational rent so that we can convince the agent to reveal his feelings.

3.

a) The first best optimal levels can be found by solving:

$$\max_{x,t} \; 2x - \alpha t$$
$$s.t. \; t - \theta x^2 \geq 0$$

It is easy to see that the constraint must be satisfied with equality. Otherwise, the social planner could increase x, the constraint would still be satisfied, and the government's objective function would yield a higher value. Hence, the problem becomes:

$$\max_{x,t} \; 2x - \alpha\theta x^2$$

The FOC yields:

$$x^*(\theta) = \frac{1}{\alpha\theta}$$

Implying a transfer of:

$$t^*(\theta) = \frac{1}{\alpha^2\theta}$$

b) In this situation, the government will maximize the expected value of its objective function (over θ) subject to the truth-telling and participation constraints. Hence, the problem becomes:

$$\max_{x(\theta),t(\theta)} \int_1^2 \left[2x(\theta) - \alpha t(\theta)\right] d\theta$$
$$s.t. \quad t(\theta) - \theta x(\theta)^2 \geq 0, \; \forall \theta \in [1,2]$$
$$t(\theta) - \theta x(\theta)^2 \geq t(\widehat{\theta}) - \theta x(\widehat{\theta})^2, \; \forall \theta, \widehat{\theta} \in [1,2]$$

Now, let $\pi(\theta, \theta) = t(\theta) - \theta x(\theta)^2$. Differentiating w.r.t. θ yields:

$$\frac{d\pi}{d\theta}(\theta,\theta) = \frac{dt}{d\theta}(\theta) - x(\theta)^2 - 2\theta x(\theta)\frac{dx}{d\theta}$$

Which becomes, after applying the envelope theorem (or incorporating the FOC):

$$\frac{d\pi}{d\theta}(\theta,\theta) = -x(\theta)^2 < 0$$

Hence, the "worst type" is $\overline{\theta}$. Since $\frac{d\pi}{dx} = -2\theta x$ and $\frac{d\pi}{dt} = 1$, we have that $\frac{d}{d\theta}\left(\frac{d\pi/dx}{d\pi/dt}\right) = -2x < 0$, and CS^- holds. This condition implies that any implementable contract must have x be a non-increasing function of type.

Integrating both sides of the previous equation from θ to $\overline{\theta}$, we obtain:

$$\pi(\theta) = \pi(\overline{\theta}) + \int_\theta^2 x(\widetilde{\theta})^2 d\widetilde{\theta}$$

$$\Leftrightarrow \quad t(\theta) = \theta x(\theta)^2 + \int_\theta^2 x(\widetilde{\theta})^2 d\widetilde{\theta}$$

Where we used the fact that the IR is binding for the "worst type", i.e., $\pi(\overline{\theta}) = 0$.

We can write the principal's problem as:

$$\max_{x(\theta),t(\theta)} \int_1^2 \left[2x(\theta) - \alpha\left(\theta x(\theta)^2 + \int_\theta^2 x(\widetilde{\theta})^2 d\widetilde{\theta}\right)\right] d\theta$$

Let us now compute $\int_1^2\int_\theta^2 x(\widetilde{\theta})^2 d\widetilde{\theta} d\theta$. Integration by parts yields:

$$\begin{aligned}
\int_1^2\int_\theta^2 x(\widetilde{\theta})^2 d\widetilde{\theta} d\theta &= \left[(\theta-1)\int_\theta^2 x(\theta)^2 d\theta\right]_1^2 + \int_1^2 (\theta-1)x(\theta)^2 d\theta = \\
&= 0 + \int_1^2 (\theta-1)x(\theta)^2 d\theta = \\
&= \int_1^2 (\theta-1)x(\theta)^2 d\theta
\end{aligned}$$

And so our problem becomes:

$$\max_{x(\theta),t(\theta)} \int_1^2 \left[2x(\theta) - \alpha(2\theta-1)x(\theta)^2\right] d\theta$$

Pointwise differentiation yields:

$$x(\theta) = \frac{1}{\alpha(2\theta-1)}$$

Hence, the transfer is:

$$\begin{aligned} t(\theta) &= \theta x(\theta)^2 + \int_\theta^2 x(\widetilde{\theta})^2 d\widetilde{\theta} = \\ &= \frac{\theta}{\alpha^2(2\theta-1)^2} + \int_\theta^2 \frac{1}{\alpha^2(2\widetilde{\theta}-1)^2} d\widetilde{\theta} = \\ &= \frac{\theta}{\alpha^2(2\theta-1)^2} - \frac{1}{2\alpha^2(4-1)} + \frac{1}{2\alpha^2(2\theta-1)} = \\ &= \frac{\theta}{\alpha^2(2\theta-1)^2} - \frac{1}{6\alpha^2} + \frac{1}{2\alpha^2(2\theta-1)} \end{aligned}$$

c) In this case, the government solves:

$$\begin{aligned} &\max_{x(\theta),t(\theta)} \int_1^{\overline{\theta}} [2x(\theta) - \alpha t(\theta)]\, d\theta \\ s.t.\quad & t(\theta) - \theta x(\theta)^2 \geq 0,\ \forall \theta \in [1, \overline{\theta}] \\ & t(\theta) - \theta x(\theta)^2 \geq t(\widehat{\theta}) - \theta x(\widehat{\theta})^2,\ \forall \theta, \widehat{\theta} \in [1, \overline{\theta}] \end{aligned}$$

Using the same steps as in *b*), it is easy to see that this problem can be re-written as:

$$\max_{x(\theta),t(\theta)} \int_1^{\overline{\theta}} \left[2x(\theta) - \alpha(2\theta-1)x(\theta)^2\right] d\theta,\ \overline{\theta} \in [1,2]$$

Pointwise differentiation yields:

$$x(\theta) = \frac{1}{\alpha(2\theta-1)}$$

And the transfer is:

$$\begin{aligned} t(\theta) &= \theta x(\theta)^2 + \int_\theta^{\overline{\theta}} x(\widetilde{\theta})^2 d\widetilde{\theta} = \\ &= \frac{\theta}{\alpha^2(2\theta-1)^2} - \frac{1}{2\alpha^2(2\overline{\theta}-1)} + \frac{1}{2\alpha^2(2\theta-1)} \end{aligned}$$

The utility of the government is:

$$B(\overline{\theta}) = \int_1^{\overline{\theta}} \left[\frac{2}{\alpha(2\theta-1)} - \frac{\theta}{\alpha(2\theta-1)^2} + \frac{1}{2\alpha(2\overline{\theta}-1)} - \frac{1}{2\alpha(2\theta-1)}\right] d\theta$$

The derivative of this function is positive and so the optimum is at $\overline{\theta} = 2$.

4.

a) Let μ denote the n-dimensional vector of the agents' valuations where each μ_i follows a distribution F on $[0,1]$. Let $T_i(\mu)$ denote the (always non-negative) amount of time that agent i will need to invest when the vector of types is μ and let $P_i(\mu)$ denote the decision function (probability of the funding going to agent i).

Let $p_i(\mu_i) = E_{\mu_{-i}}(P(\mu))$ and $t_i(\mu_i) = E_{\mu_{-i}}(T_i(\mu))$.

The (interim) utility of an agent i is then $u_i = p_i\mu_i - t_i$

(i) IC

For agent i's IC, from $U_i(\widehat{\mu}_i, \mu_i) = -t_i(\widehat{\mu}_i) + \mu_i p_i(\widehat{\mu}_i)$ we can write the value function $U_i(\mu_i) \equiv U_i(\mu_i, \mu_i)$. Using the value function and applying the envelope theorem (or just incorporating the FOC), we have that $\frac{dU_i}{d\mu_i}(\mu_i) = \frac{\partial u_i}{\partial \mu_i}(\mu_i)$ and

$$\frac{dU_i}{d\mu_i}(\mu_i) = p_i(\mu_i).$$

Since $\frac{d}{d\mu_i}\left(\frac{\frac{\partial u_i}{\partial p_i}}{\frac{\partial u_i}{\partial(-t_i)}}\right) = 1 > 0$, CS^+ holds. (Notice that t_i enters the agent's utility function with a negative sign and therefore we need to take the derivative with respect to $-t_i$ in order to apply the theorems directly).

Therefore, we need $p_i(\mu_i)$ to be nondecreasing in any implementable contract.

$$IC \iff \begin{cases} \frac{dU_i}{d\mu_i}(\mu_i) = p_i(\mu_i) \\ p_i(\mu_i) \text{ nondecreasing} \end{cases}$$

We can now rewrite the utility of agent i as the sum of the utility of the "worst type" and an integral. Integrating $\frac{dU_i}{d\mu_i}(\mu_i) = p_i(\mu_i)$ from $\underline{\mu_i} = 0$ to μ_i

yields $U_i(\mu_i) = U_i(0) + \int_0^{\mu_i} p_i(\widetilde{\mu}_i)d\widetilde{\mu}_i$. Since $U_i(\mu_i) = -t_i(\mu_i) + \mu_i p_i(\mu_i)$, we can write $t_i(\mu_i) = \mu_i p_i(\mu_i) - U_i(0) - \int_0^{\mu_i} p_i(\widetilde{\mu}_i)d\widetilde{\mu}_i$.

b) IR just requires that $U_i(0) \geq 0$. Notice however that because time is non-negative, there is an upper bound of 0 for the utility of an agent with a valuation of 0. Therefore, the Ministry must set the contract for type 0 such that $U_i(0) = 0$.

The Ministry wants to maximize $\sum_i E_{\mu_i}[\int_0^{\mu_i} p_i(\widetilde{\mu}_i)d\widetilde{\mu}_i]$ (subject to the monotonicity constraints).

Since

$$E_{\mu_i}\left[\int_0^{\mu_i} p_i(\widetilde{\mu}_i)d\widetilde{\mu}_i\right] = \int_0^1 \int_0^{\mu_i} p_i(\widetilde{\mu}_i)d\widetilde{\mu}_i f(\mu_i)d\mu_i,$$

integration by parts yields

$$E_{\mu_i}\left[\int_0^{\mu_i} p_i(\widetilde{\mu}_i)d\widetilde{\mu}_i\right] = E_{\mu_i}\left[p_i(\mu_i).\frac{1-F(\mu_i)}{f(\mu_i)}\right]$$

Ignoring the monotonicity constraint, the simplified problem is then:

$$\max_{\{P_i(\mu)\}} \sum_i E_{\mu_i}\left[p_i(\mu_i).\frac{1-F(\mu_i)}{f(\mu_i)}\right] = \max_{\{P_i(\mu)\}} \sum_i E_{\mu}\left[P_i(\mu).\frac{1-F(\mu_i)}{f(\mu_i)}\right]$$

Let $J(\mu_i) \equiv .\frac{1-F(\mu_i)}{f(\mu_i)}$.Given the assumption, $J(.)$ is strictly decreasing (and it is always non-negative). Ignoring the monotonicity constraint, the optimal mechanism would be the one that sets: $P_i(\mu) = \begin{cases} 1 & \text{if } \mu_i = \min\{\mu_1, ..., \mu_n\} \\ 0 & \text{otherwise} \end{cases}$

However, the monotonicity constraint is not satisfied by this mechanism. Incorporating the monotonicity constraint, we have that (i) it is always optimal to set the sum of the probabilities equal to 1; (ii) $P_i(\mu) \geq P_j(\mu)$ whenever $\mu_i \geq \mu_j$. The optimal mechanism (that maximizes the Ministry's expected utility) must then set $P_i(\mu) = P_j(\mu)$ whenever $\mu_i \geq \mu_j$ and therefore $P_i(\mu) = P_j(\mu)$ for all i, j.

The optimal mechanism will then set $P_i(\mu) = \frac{1}{N}$ for all i and $t_i(\mu_i) = \frac{\mu_i}{N} - \int_0^{\mu_i} \frac{1}{N} d\widetilde{\mu}_i = 0$ (and the grants will be allocated randomly).

5.

a) A single firm; government has coercive power (i.e. we do not need to worry about IR).

Show that the socially optimal amount of pollution $x^*(\theta)$ can be obtained by giving the firm a transfer $k - D(x)$.

The social welfare function is, up to a constant, $-D(x) - C(x, \theta)$ (i.e., the government cares about the damages to society and also about the firm´s profits)

The socially optimal $x^*(\theta)$ comes from the government problem:

$$\max_x -D(x) - C(x, \theta)$$

FOC:

$$D'(x) = -C'(x, \theta) \tag{1}$$

which implicity defines $x^*(\theta)$.

(The SOC is also satisfied since $-D'(x) - C_{xx}(.,.) \leq 0$)

If the government gives the firm a transfer $t(x) = k - D(x)$,the firm solves the problem:

$$\max_x t(x) - C(x, \theta) = \max_x k - D(x) - C(x, \theta)$$

FOC:

$$D'(x) = -C'(x, \theta)$$

where clearly $x = x^*(\theta)$ is the solution.

Therefore, the firm will internalize the externality and choose the socially optimal level of pollution (just like in the Groves case, the government induces the firm to solve the social welfare problem) b)

In an A-G-V scheme, each agent is paid the expected value of the other agent´s surplus, conditional on his own report. In this example, since there is an additional element in the social welfare function (the damages caused by pollution), we need to include that as well in the transfer so that each agent will internalize the social surplus and choose the socially optimal x_i^*.

So we set

$$t_i\left(\widehat{\theta}\right)=E_{\theta_{-i}}\left[\sum_{j\neq i}-C_j\left(x_j^*\left(\widehat{\theta}_i,\theta_{-i}\right),\theta_j\right)-D\left(x^*\left(\widehat{\theta}_i,\theta_{-i}\right)\right)\right]+\tau_i\left(\widehat{\theta}_{-i}\right)$$

where $\tau_i\left(\widehat{\theta}_{-i}\right)$ is constant in $\widehat{\theta}_i$ and is chosen so that the budject is balanced.

Clearly $\widehat{\theta}_i=\theta_i$ maximizes

$$E_{\theta_{-i}}\left[\sum_{j\neq i}-C_j\left(x_j^*\left(\widehat{\theta}_i,\theta_{-i}\right),\theta_j\right)-D\left(x^*\left(\widehat{\theta}_i,\theta_{-i}\right)\right)-C_i\left(x_i^*\left(\widehat{\theta}_i,\theta_{-i}\right),\theta_i\right)\right]$$

and the mechanism is incentive compatible.

For budget balance, we share the expected externality we give to each agent among all other agents, i.e.,

$$\tau_i\left(\widehat{\theta}_{-i}\right)=-\frac{1}{I-1}\sum_{j\neq i}E_{\theta_{-j}}\left(\sum_{k\neq j}-C_k\left(x_k^*\left(\widehat{\theta}_j,\theta_{-j}\right),\theta_k\right)-D\left(x^*\left(\widehat{\theta}_j,\theta_{-j}\right)\right)\right)$$

So

$$\begin{aligned}t_i\left(\widehat{\theta}\right) &= E_{\theta_{-i}}\left(\sum_{j\neq i}-C_j\left(x_j^*\left(\widehat{\theta}_i,\theta_{-i}\right),\theta_j\right)-D\left(x^*\left(\widehat{\theta}_i,\theta_{-i}\right)\right)\right)\\ &\quad -\frac{1}{I-1}\sum_{j\neq i}E_{\theta_{-j}}\left(\sum_{k\neq j}-C_k\left(x_k^*\left(\widehat{\theta}_j,\theta_{-j}\right),\theta_k\right)-D\left(x^*\left(\widehat{\theta}_j,\theta_{-j}\right)\right)\right)\end{aligned}$$

6.

(a) Player i´s payoff function is:

$$u_i(b_i, b_{-i}, \theta_i, \theta_{-i}) = \begin{cases} \theta_i - b_i & i \text{ submits winning bid} \\ -b_i & i \text{ submits losing bid} \end{cases}$$

The set of strategies ($b_j(\theta_j)$ $\forall j = 1, ..., I$) constitutes a BNE here, if for each $v_i \in [0,1]$, $b_i(v_i)$ solves:

$$\max_{b_i} (\theta_i - b_i) \Pr[b_i > b_j(\theta_j)]^{I-1} + (-b_i)\left[1 - \Pr[b_i > b_j(\theta_j)]^{I-1}\right]$$

$$\Leftrightarrow \max_{b_i} -b_i + \theta_i \Pr[b_i > b_j(\theta_j)]^{I-1}$$

To solve for a BNE, suppose that player j adopts the strategy $b(.)$ and assume that $b(.)$ is strictly increasing and differentiable. Then for a given realization of θ_i , player *i's* optimal bid solves:

$$\max_{b_i} -b_i + \theta_i \Pr[b_i > b_j(\theta_j)]^{I-1}$$

Let $b_i^{-1}(b_j) = b^{-1}(b(\theta_j)) = \theta_j$ the valuation that player j must have in order to be bidding b_j. Since $\theta_j \sim U[0,1]$ we have:

$$\begin{aligned} & -b_i + \theta_i \Pr[b_i > b_j(\theta_j)]^{I-1} \\ = & -b_i + \theta_i \Pr\left[b^{-1}(b_i) > b^{-1}(b(\theta_j))\right]^{I-1} \\ = & -b_i + \theta_i \left[\frac{b^{-1}(b_i)}{1-0}\right]^{I-1} \end{aligned}$$

Thus, the first order condition for player *i's* optimization problem is:

$$-1 + \theta_i (I-1)\left[\frac{db^{-1}(b_i)}{db_i}\right]\left[b^{-1}(b_i)\right]^{I-2} = 0$$

The first order condition (III) is an implicit equation for bidder *i's* best response to the strategy $b(.)$ played by bidder j, given that bidder *i's* valuation has been realized as θ_i . If we are looking for a symmetric BNE, we require that both bidders play the same strategy in equilibrium. Since, therefore, bidder j plays the strategy $b(.)$, this must be also playedby bidder i, in equilibrium. Hence, we require that $b(.)$ is player *i's* best response to $b(.)$ by player j. In other words, $b(_i)$ must satisfy the first order condition (II): that is, for each of bidder *i's* positive valuations, she does not wish to deviate from bidding according to the schedule $b(.)$, given that player j bids according to the same schedule.

To impose this requirement, we substitute $b_i = b(\theta_i)$ into (iii):

$$\begin{aligned} -1 + \theta_i (I-1)\left[\frac{db^{-1}(b_i)}{db_i}\right]\left[b^{-1}(b_i)\right]^{I-2} &= 0 \Leftrightarrow \theta_i (I-1)\left[\frac{d\theta_i}{db_i}\right]\theta_i^{I-2} = 1 \\ \theta_i^{I-1}(I-1)\left[\frac{db_i}{d\theta_i}\right] &= 1 \end{aligned}$$

Our last equation must be viewed as a first-order differential equation that the function $b(.)$ must satisfy. Clearly, however, if this is to be satisfy for any values of θ_i $\forall i$, it should be so for $\theta_i = \theta$ $\forall i$. We now have:

$$\frac{\theta^{I-1}}{b'(\theta)}(I-1) = 1 \Leftrightarrow b'(\theta) = (I-1)\theta^{I-1} \Leftrightarrow b(\theta) = \frac{I-1}{I}\theta^I + c$$

To eliminate c we need a boundary condition. Fortunately, simple economic reasoning provides us with one: no player should bid more than his/her valuation. Thus, we require $b(\theta_i) \leq \theta_i$ $\forall \theta_i \in [0,1]$. In particular, we require $b(0) \leq 0$. Since bids are constrained to be non-negative, this implies that $b(0) = 0$. Hence, $c = 0$ and our proposed BNE solution is that each bidder submits a bid according to the schedule:

$$b(\theta_i) = \frac{I-1}{I}\theta_i^I$$

(b) Using the Revenue-Equivalence theorem, we note that:

1. Both auctions can be viewed as incentive-compatible, direct-selling mechanisms.

2. In both auctions, the probability assignment function is the same since the object is assigned, in equilibrium, to the player with the highest valuation.

This is due the fact that, in both auctions, the players. equilibrium bidding strategies are strictly increasing in the players own valuations.

3. In both auctions, a bidder with zero valuation receives an equilibrium expected payoff of zero. Therefore, he is clearly indifferent between the two auction mechanisms.

(1)-(3) suffice for the theorem to apply. Consequently, the expected revenue to the seller ought to be the same between the sealed-bid second price auction and the first-price all-pay auction.

(c) Optimal asymmetric auction

- risk neutrality and independent private values

$\theta_1 \tilde{} U[0,10]; \theta_2 \tilde{} U[0,1]; \theta_3 \tilde{} U[0,1]$

We can apply the same steps as in the lecture notes to get the expected revenue to seller:

$\int_0^{10} \partial\theta_1 \int_0^1 \partial\theta_2 \int_0^1 \partial\theta_3 [P_1(\theta_1,\theta_2,\theta_3)J_1(\theta_1) + P_2(\theta_1,\theta_2,\theta_3)J_2(\theta_2) + P_3(\theta_1,\theta_2,\theta_3)J_3(\theta_3)]\frac{1}{10}$

where the virtual valuations are:

$J_1(\theta_1) = 2\theta_1 - 10$; $J_2(\theta_2) = 2\theta_1 - 1$; $J_3(\theta_3) = 2\theta_3 - 1$;

All these are increasing in θ_i so the problem is regular $\rightarrow$ the optimal auction is characterized by:

Which gives:

$P_1(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_1 > 5$ and $\theta_1 > 4.5 + \theta_2$ and $\theta_1 > 4.5 + \theta_3$

$P_2(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_2 > 0.5$ and $\theta_2 > \theta_3$ and $\theta_2 > \theta_1 - 4.5$

$P_3(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_3 > 0.5$ and $\theta_3 > \theta_2$ and $\theta_3 > \theta_1 - 4.5$

We know from class that

$T_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} X_i(\widetilde{\theta}_i)d\widetilde{\theta}_i - \theta_i X_i(\theta_i)$

Rewriting it with the notation of the exercise ($p_i = X_i$ and $P_i = x_i$) and using the fact that the utility of the worst type must be zero, we get (using integration by parts):

$T_i(\theta_i) = \int_{0_i}^{\theta_i} p_i(\widetilde{\theta}_i)d\widetilde{\theta}_i - \theta_i p_i(\theta_i) = -\int_{0_i}^{\theta_i} \widetilde{\theta}_i dp(\widetilde{\theta}_i)$

So the agent expects to pay $t_i(\theta_i) = \int_{0_i}^{\theta_i} \widetilde{\theta}_i dp(\widetilde{\theta}_i)$

(where $p_i(\theta_i) = E_{\theta_{-i}}[P_i(\theta)]$).

One way to replicate this optimal auction by a standard-like auction:

- consider a Vickrey second price auction with a reserve price of 5 but where bidders 2 and 3 get a rebate of 4.5 if they win (this means that bidder 1 end up paying more than 5 if he wins). In this auction it is a dominant strategy to bid one´s valuation (+4.5 for bidders 2 and 3) $\rightarrow$ probabilities of winning and expected payment correspond to the ones found above.

Notice: An assumption all along is that the auctioners knows the distributions of types across bidders and who is who Bidders 2 and 3 receive preferential treatment here. This policy has a cost: sometimes the object will be awarded to someone else than the highest valuation bidder (non-efficiency) and the auctioneer will receive a relatively low payment in those cases.

However this bias against 1 forces him to bid higher than otherwise. The optimal auction strikes the best compromise between these two effects.

(d) $\theta_1 \tilde{} U[1,11]$

As above, we get:

$J_1(\theta_1) = 2\theta_1 - 11$; $J_2(\theta_2) = 2\theta_2 - 1$; $J_3(\theta_3) = 2\theta_3 - 1$

Which gives:

$P_1(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_1 > 5.5$ and $\theta_1 > 5 + \theta_2$ and $\theta_1 > 5 + \theta_3$

$P_2(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_2 > 0.5$ and $\theta_2 > \theta_3$ and $\theta_2 > \theta_1 - 5$

$P_3(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_3 > 0.5$ and $\theta_3 > \theta_2$ and $\theta_3 > \theta_1 - 5$

A similar 2nd price auction with minimum bid of 5.5 and rebate of 5 for bidders 2 and 3 is optimal

And this even though the auctioneer a priori knows that bidder 1's valuation is always greater than bidders 2 and 3's valuations.