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1. Consider a pure exchange economy with two infinitely-lived agents, $i = 1, 2$, and time indexed by $t = 0, 1, 2, \ldots$. There is a single consumption good in each period. The state of the world at time t is denoted by $s_t \in S$, and a history up to time t is denoted $s^t = (s_0, s_1, \ldots, s_t) \in S^{t+1}$. Each history s^t occurs with probability $\pi_t(s^t)$, where $\pi_0(s^0) = 1$, and for $t \geq 1$:

$$\pi_t(s^t) = \pi_{t-1}(s^{t-1}) \cdot \pi(s_t | s^{t-1}),$$

with $\pi(s_t|s^{t-1})$ being the conditional probability of state s_t given history s^{t-1}. Each agent receives an endowment stream $\{e_t^i(s^t)\}_{t=0}^{\infty}$ and has preferences over stochastic consumption sequences:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t^i(s^t)) \right] = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) u(c_t^i(s^t)),$$

where $0 < \beta < 1$ and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$. Markets are complete, and we will compare two different trading environments:

- A **complete set of Arrow-Debreu securities** traded at date 0.
- A **sequential equilibrium** with one-period Arrow securities traded at each date contingent on next period's state.

Arrow-Debreu Equilibrium

(a) Define a competitive equilibrium with a complete set of Arrow-Debreu securities traded at $t = 0$.

(b) Formulate the social planner's problem to compute the Pareto optimal allocation:

$$\max_{\{c_t^1(s^t), c_t^2(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \pi_t(s^t) \beta^t \left[\lambda u(c_t^1(s^t)) + (1-\lambda) u(c_t^2(s^t)) \right]$$

subject to:

$$c_t^1(s^t) + c_t^2(s^t) \leq e_t^1(s^t) + e_t^2(s^t) \quad \text{for all } t, s^t.$$

(c) Show that the Arrow-Debreu equilibrium allocation coincides with the planner's allocation for an appropriate choice of λ.

(d) Derive the equilibrium prices, $q_t(s^t)$, and allocations that support the competitive equilibrium.

Sequential Equilibrium with Arrow Securities

(e) Define a sequential competitive equilibrium with one-period state-contingent Arrow securities traded at each date.

(f) Write down the household's dynamic program, given initial state s^0.

(g) Derive the Euler and market clearing conditions under this sequential framework.

(h) Show that the same consumption allocation as in the Arrow-Debreu equilibrium can be supported by appropriate sequential prices $Q(s_{t+1}|s^t)$ and asset holdings.

Incomplete Markets: Risk-Free Bond in Zero Net Supply.

Now suppose that instead of complete markets, the only financial asset available is a one-period risk-free bond. The bond pays one unit of the consumption good in the next period, regardless of the realized state. It is traded at each date t at price $\tilde{q}_t(s^t)$ and is in zero net supply.

(i) Define the sequential competitive equilibrium under this market structure.

(j) Write the household's dynamic programming problem. Let $b_t^i(s^t)$ denote agent i's holdings of the bond purchased at t in state s^t.

(k) Derive the first-order conditions and explain how the bond price $\tilde{q}_t(s^t)$ is determined in equilibrium.

(l) Impose the bond market clearing condition:

$$\sum_{i=1}^{2} b_t^i(s^t) = 0, \quad \forall t, s^t.$$

(m) Compare the equilibrium consumption allocations in this setting to those under complete markets. In particular, discuss: Consumption smoothing possibilities for the agents and whether efficient risk-sharing is achieved.

2. Consider an exchange economy with two agents $i = 1, 2$. The endowments of both agents are stochastic and given by:

$$e^1(s^t) = e^2(s^t) = s_t,$$

where $s^t = (s_0, \ldots, s_t)$ denotes a history up to date t and $s_t \in S = \{1, 2\}$ is the realization of the state at time t. Thus, at each date t and history s^t, both agents receive an endowment of s_t units of the consumption good. Let $P(s^t)$ denote the probability of history $s^t \in S^t$, where S^t is the set of all possible t-period histories. The stochastic process $\{s_t\}$ is assumed to be Markovian with time-invariant transition probability $\pi(s'|s)$, where s is the current state and s' is the next period's state. Preferences of both agents are given by:

$$U^i(c) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t u(c_t^i(s^t)) P(s^t), \quad \text{for } i = 1, 2,$$

where $0 < \beta < 1$ and $u(\cdot)$ is a strictly increasing, strictly concave, and differentiable period utility function.

(a) Define and calculate (i.e., write down explicitly) a competitive equilibrium with time-0 (Arrow-Debreu) markets for this economy, assuming $s_0 = 1$.

Now consider the same exchange economy, with the same agents, preferences, and endowments. The only difference is the market structure. In the present case, instead of a complete set of time-0 markets for all future contingencies, the economy has sequential markets with a complete set of **one-period Arrow securities** traded in each period. At each date t, there are two Arrow securities:

- One pays 1 unit of the consumption good at $t+1$ if $s_{t+1} = 1$ and 0 otherwise.
- The other pays 1 unit of the consumption good at $t+1$ if $s_{t+1} = 2$ and 0 otherwise.

Let $Q(s_{t+1}|s^t)$ denote the price at history s^t of the Arrow security that pays off when state s_{t+1} is realized.

(b) Define and calculate (i.e., write down explicitly) a competitive equilibrium with sequential markets for this economy, assuming $s_0 = 1$.

3. Consider a production economy with one representative consumer and one firm. The model has two periods: period 0 and period 1. In each period, the firm produces a consumption good using labor as the only input. The production function is linear in labor and depends on the productivity shock s_t:

- Period 0: $y = s_0 F(L) = s_0 L$, where $s_0 = 1$.

- Period 1: $y = s_1 F(L) = s_1 L$, where $s_1 \in S = \{1, 2\}$.

The firm chooses labor demand in each period to maximize profits, taking wages as given. The representative consumer supplies labor and consumes in both periods. The consumer's utility is given by:

$$U = u(c(s_0), \ell(s_0))P(s_0) + \sum_{s_1 \in S} u(c(s_0, s_1), \ell(s_0, s_1))P(s_0, s_1),$$

where the period utility function is:

$$u(c, \ell) = \log c - \frac{\ell^2}{2},$$

and $P(s_0) = 1$, $P(s_0, s_1) = \frac{1}{2}$ for each $s_1 \in S$.

(a) Define a competitive equilibrium with time-0 (Arrow-Debreu) markets in this economy.

(b) Calculate a competitive equilibrium with time-0 markets for this economy. In particular:

4. Consider the following deterministic income fluctuation problem:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$\begin{aligned} c_t + a_{t+1} &= (1 + r)a_t + y_t, \quad \text{for all } t \geq 0, \\ a_{t+1} &\geq -\bar{a} \leq 0, \\ a_0 &= 0, \end{aligned}$$

where:

- $u' > 0$, $u'' < 0$ (strictly increasing and concave utility),
- $\beta(1 + r) = 1$,
- $\bar{a} < \infty$ is the borrowing limit,
- The endowment process is growing: $y_t = (1 + g)^t$, with $0 < g < r$.

(a) Solve for the optimal consumption and wealth path when $\bar{a} = 0$ (i.e., no borrowing is allowed).

(b) Now assume the borrowing constraint takes the form:

$$a_{t+1}/y_{t+1} \geq -\bar{a},$$

Solve for the minimum value of $\bar{a}$ that allows the consumer to maintain constant consumption over time (i.e., $c_t = c$ for all t).

5. Consider an economy populated by a continuum (measure 1) of ex-ante identical consumers. Each consumer receives stochastic labor income in the form of efficiency units of labor, denoted ε, which is i.i.d. across time and across agents, has non-negative support, has mean μ. Consumers can save in a risk-free, non-contingent bond $a \geq 0$ (no borrowing is allowed), and they make consumption and savings decisions to maximize lifetime utility. Assume that the optimal decision rule for next-period assets is linear and given by:

$$a' = (1 + r)a + \gamma w \varepsilon,$$

where $0 < \gamma < 1$ is a constant, and r and w are the equilibrium interest and wage rates, respectively. There is a representative firm that hires labor and capital in competitive markets at prices w and r, and operates a constant-returns-to-scale production technology:

$$Y = AK^{\alpha}L^{1-\alpha} + (1 - \delta)K,$$

where $A > 0$ is total factor productivity, $0 < \alpha < 1$ is capital's share, $0 < \delta < 1$ is the depreciation rate of capital.

(a) Show that a stationary distribution of consumers over asset holdings and income shocks exists only if the interest rate satisfies $r < 0$.

(b) For all values of $r < 0$, solve and plot: the stationary supply of assets, i.e., $E(a)$, and the demand for capital by the representative firm, $K(r)$.

(c) Solve for the equilibrium interest rate r^* and wage rate w^*.

(d) Discuss what happens to long run equilibrium GDP, interest and wage rate if consumer want to save more (γ increases) or if productivity of the firms decline (A goes down).

6. Consider a Aiyagari (1993) economy. Each consumer faces idiosyncratic labor income risk. Agents can save using a risk-free asset, but borrowing is restricted. The production side features a representative firm operating under constant returns to scale. Households maximize expected lifetime utility:

$$\max_{\{c_t, a_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

subject to:

$$c_t + a_{t+1} \leq w z_t + (1+r) a_t \tag{2}$$

$$a_{t+1} \geq \underline{a} \tag{3}$$

$$z_{t+1} \sim \Pi(z_{t+1} | z_t) \tag{4}$$

where: c_t is consumption at time t, a_t are asset holdings, $z_t \in \{z_1, z_2, ..., z_n\}$ is idiosyncratic labor productivity with stationary distribution π, w is the wage rate, r the interest rate; $\beta \in (0,1)$ is a time discount factor, $u(\cdot)$ is a CRRA utility function, $\underline{a} = 0$ is a no-borrowing constraint, and Π is a Markov transition probability matrix for z_t. A representative firm chooses capital K and labor L to maximize profits:

$$\max_{K,L} F(K,L) - (r+\delta)K - wL \tag{5}$$

where δ is the depreciation rate of capital. Assuming a Cobb-Douglas production function, with $\alpha \in (0,1)$:

$$F(K,L) = K^{\alpha} L^{1-\alpha}. \tag{6}$$

(a) Define a stationary competitive equilibrium of this economy.

(b) Write the household's problem recursively as a dynamic programming problem. Clearly define the state and control variables, and write the Bellman equation.

(c) Derive the intertemporal Euler equation from the household's problem. Discuss how the borrowing constraint affects the Euler equation and under what conditions the constraint binds.

(d) Explain how the law of motion for assets and the Markov process for z jointly determine the evolution of the invariant distribution $\mu(a,z)$, and write the fixed-point condition that must be satisfied by μ. Discuss what it describes in the economy.

(e) Suppose the borrowing constraint $\underline{a}$ is loosened marginally. Use comparative statics or envelope theorem arguments to predict the effect on the household's optimal savings and aggregate capital.