Microeconometrics
Ordinal Models

Pedro Portugal

Nova School of Business and Economics

Spring 2025

Examples of Ordinal Variables

- Income Levels
  - Low, Average, High
- Evaluations, Rankings
  - First, Second, Third, ...
- Levels of Agreement
- Football match outcome
  - Win, Draw, and Loss
- Preferences
  - Example: Nationwide choice of university

Ordered Probit

- Let y denote a response variable that can take the values $0, 1, 2, ..., J$
- The starting point is an index model, with single latent variable

$$y_i^* = \mathbf{x}_i' \beta + u_i$$

where $u|\mathbf{x} \sim \mathcal{N}(0,1)$

- In general for an m-alternative **ordered model** we define:

$$y_i = j \qquad \text{if } \alpha_{j-1} < y_i^* \leq \alpha_j$$

- Therefore,

$$\begin{aligned} y_i &= 0 && \text{if } y_i^* \leq \alpha_1 \\ y_i &= 1 && \text{if } \alpha_1 < y_i^* \leq \alpha_2 \\ y_i &= 2 && \text{if } \alpha_2 < y_i^* \leq \alpha_3 \\ &\dots \\ y_i &= J && \text{if } y_i^* \geq \alpha_J \end{aligned}$$

Ordered Probit

Then,

$$Pr[y_i = 0|\mathbf{x}] = Pr[y_i^* \leq \alpha_1] = \Phi(\alpha_1 - \mathbf{x}_i'\beta)$$
$$Pr[y_i = 1|\mathbf{x}] = Pr[\alpha_1 < y_i^* \leq \alpha_2] = \Phi(\alpha_2 - \mathbf{x}_i'\beta) - \Phi(\alpha_1 - \mathbf{x}_i'\beta)$$
$$\ldots$$
$$Pr[y_i = J-1|\mathbf{x}] = Pr[\alpha_{J-1} < y_i^* \leq \alpha_J] = \Phi(\alpha_J - \mathbf{x}_i'\beta) - \Phi(\alpha_{J-1} - \mathbf{x}_i'\beta)$$
$$Pr[y_i = J|\mathbf{x}] = Pr[y_i^* > \alpha_J] = 1 - \Phi(\alpha_J - \mathbf{x}_i'\beta)$$

- where $\Phi(.)$ is the standard normal cdf of u_i

And the log-likelihood function:

$$\log L(\alpha, \beta) = \sum_{i=1}^{N} \mathbf{1}[y_i = 0] \log[\Phi(\alpha_1 - \mathbf{x}_i'\beta)] + \mathbf{1}[y_i = 1] \log[\Phi(\alpha_2 - \mathbf{x}_i'\beta) - \Phi(\alpha_1 - \mathbf{x}_i'\beta)] +$$
$$+ \ldots + \mathbf{1}[y_i = J] \log[1 - \Phi(\alpha_J - \mathbf{x}_J'\beta)]$$

Marginal Effects

- The sign of the regression parameters β can be immediately interpreted as determining whether or not the latent variable y^* increases with the regressor

- For marginal effects in the probabilities

$$\frac{\partial Pr[y_i = 0]}{\partial x_k} = -\beta_k \phi(\alpha_1 - \mathbf{x}_i' \beta)$$

$$\frac{\partial Pr[y_i = J]}{\partial x_k} = \beta_k \phi(\alpha_J - \mathbf{x}_i' \beta)$$

$$\frac{\partial Pr[y_i = j]}{\partial x_k} = \beta_k [\phi(\alpha_{j-1} - \mathbf{x}_i' \beta) - \phi(\alpha_j - \mathbf{x}_i' \beta)]$$

Examples

- Base de dados sobre o percurso de licenciados (College Graduates Career Data Base)
  - Occupational satisfaction
  - Firm size
    - Small, medium, large
  - Changes after graduation
    - Wage raise
    - Promotion
    - No change

Ordered Logit

- For the **ordered logit model** u is logistic distributed with

$$\Lambda(z) = e^z/(1+e^z)$$

- Then,

$$Pr[y_i = 0|\mathbf{x}] = Pr[y_i^* \leq \alpha_1] = Pr[\mathbf{x}_i'\beta + u_i \leq \alpha_1|\mathbf{x}] = \Lambda(\alpha_1 - \mathbf{x}_i'\beta)$$
$$Pr[y_i = 1|\mathbf{x}] = Pr[\alpha_1 < y_i^* \leq \alpha_2] = \Lambda(\alpha_2 - \mathbf{x}_i'\beta) - \Lambda(\alpha_1 - \mathbf{x}_i'\beta)$$
...
$$Pr[y_i = J-1|\mathbf{x}] = Pr[\alpha_{J-1} < y_i^* \leq \alpha_J] = \Lambda(\alpha_J - \mathbf{x}_i'\beta) - \Lambda(\alpha_{J-1} - \mathbf{x}_i'\beta)$$
$$Pr[y_i = J|\mathbf{x}] = Pr[y_i^* > \alpha_J] = 1 - \Lambda(\alpha_J - \mathbf{x}_i'\beta)$$

Grouped Data (Interval Censored Data)

- Consider that the latent variable $y^*|\mathbf{x} \sim \mathcal{N}(\mathbf{x}\beta, \sigma^2)$
- $a_1 < a_2 < ... < a_J$ and the limits of the intervals are known

Then

$$\log L(\alpha, \beta) = \sum_{i=1}^{N} \mathbf{1}[y_i = 0] \log \Phi\left(\frac{a_1 - \mathbf{x}_i'\beta}{\sigma}\right) + \mathbf{1}[y_i = 1] \log\left[\Phi\left(\frac{a_2 - \mathbf{x}_i'\beta}{\sigma}\right) - \Phi\left(\frac{a_1 - \mathbf{x}_i'\beta}{\sigma}\right)\right]$$
$$+ ... + [y_i = J] \log\left[\left(1 - \Phi\left(\frac{a_J - \mathbf{x}_i'\beta}{\sigma}\right)\right]$$

And

$$\frac{\partial y^*}{\partial x_k} = \beta_k$$

Example

- Labor earnings from the Employment Household Survey
- Impact of job finding methods on wages

Example

Equação de Regressão do Salário Mensal Líquido, Dados Agrupados,1997

Variáveis	Total	Contratos Permanentes	Contratos a Termo
Experiência no Mercado de Trabalho	0.0267	0.0273	0.018
	(0.0006)**	(0.0007)	(0.0013)
Experiência^2	-0.0004	-0.0004	-0.0003
	(0.00001)	(0.00001)	(0.00003)
Escolaridade Básica (1º ciclo)	0.0958	0.1101	0.0003
	(0.0095)	(0.0102)	(0.0256)
Escolaridade Básica (2º e 3º ciclos)	0.3052	0.3249	0.1239
	(0.0100)	(0.0108)	(0.0267)
Secundário	0.5932	0.6352	0.2952
	(0.0113)	(0.0122)	(0.0290)
Superior	1.1119	1.1456	0.816
	(0.0114)	(0.0122)	(0.0303)
Formação Profissional	0.1178	0.1306	0.043
	(0.0079)	(0.0087)	(0.0183)
Antiguidade no Posto de Trabalho	0.0127	0.0121	0.0234
	(0.0007)	(0.0008)	(0.0049)
Antiguidade^2	-0.0001	-0.0001	-0.0007
	(0.00002)	(0.00002)	(0.00038)
Homem	0.2264	0.2313	0.1996
	(0.0042)	(0.0046)	(0.0105)
Contrato a Prazo	-0.0309		
	(0.0063)		
Constante	3.5068	3.4342	3.8926
	(0.0154)	(0.0169)	(0.0348)
sigma	0.3205	0.3196	0.3067
N	31573	24300	4273
Log-Verosimilhança	-58131.5	-49042.7	-8873.6

* As regressões incluem 6 variáveis dummy sectoriais

** desvios padrões assintóticos entre parêntesiss

Ranked Data Models

- Alternatives may be ranked, especially with stated preference data
- The **rank-ordered logit model** is simple to estimate
  - Consider a four-alternative conditional logit model with alternative 2 the first choice and alternative 3 the second choice
  - Alternative 1 is chosen from all the 4 alternatives and then alternative 3 is chosen from the remaining alternatives 1, 3 and 4
  - The joint probability of these first and second choices is

$$\frac{e^{\mathbf{x}_{i2}'\beta}}{e^{\mathbf{x}_{i1}'\beta}+e^{\mathbf{x}_{i2}'\beta}+e^{\mathbf{x}_{i3}'\beta}+e^{\mathbf{x}_{i4}'\beta}} \times \frac{e^{\mathbf{x}_{i3}'\beta}}{e^{\mathbf{x}_{i1}'\beta}+e^{\mathbf{x}_{i3}'\beta}+e^{\mathbf{x}_{i4}'\beta}}$$

- Estimation is by ML given similar expressions for the other 11 joint probabilities

Bivariate and Multivariate Discrete Outcomes

- Consider bivariate discrete data (y_{1i}, y_{2i})
- For example, in a joint model of labor supply and fertility the dependent variables (y_{1i}, y_{2i}) for individual i may be $y_{1i} = 2$ if work and $y_{1i} = 1$ do not work, and $y_{2i} = 2$ if have children and $y_{2i} = 1$ if have no children
- More generally, y_1 may take values $1, ..., m_1$ and y_2 may take values $1, ..., m_2$
- For individual i define

$$p_{ijk} = Pr[y_{1i} = j, y_{2i} = k], \quad j = 1, ..., m_1 \quad k = 1, ..., m_2$$

where p_{ijk} define probabilities of mutually exclusive events and $\sum_j \sum_k p_{ijk} = 1$

Multivariate Discrete Outcomes

- Define $y_{jk} = 1$ if $(y_1 = j, y_2 = k)$ and $y_{jk} = 0$ otherwise. Then the joint density for the ith observation is

$$f(y_{1i}, y_{2i}) = \prod_{k=1}^{m_1} \prod_{j=1}^{m_2} p_{ijk}^{y_{ijk}}$$

- The log-likelihood is then $\sum_{i=1}^{N} \sum_{k=1}^{m_1} \sum_{j=1}^{m_2} y_{ijk} \ln p_{ijk}$ and estimation is by ML

- In the simplest case the two discrete dependent variables are independent and $p_{ijk} = Pr[y_{1i} = j] \times Pr[y_{2i} = k]$

Sequential Multinomial Probit

- Consider the two events "go to college" and "take a post-graduation". Then

$$Pr[y=2] = Pr[y=2|y\neq 1]\times Pr[y\neq 1] = \Phi(x_2\beta_2)(1-\Phi(x_1\beta_1))$$

- And

$$L = \sum_{i=1}^{N}\sum_{j=1}^{m} y_{ij}p_{ij}$$

- Therefore

$$L = \sum_{i=1}^{N} y_{i1}\ln\left[\Phi(x_1\beta_1)\right] + y_{i2}\ln\left[\Phi(x_2\beta_2)(1-\Phi(x_1\beta_1))\right] + \\ + y_{i3}\ln\left[1-\Phi(x_1\beta_1)-\Phi(x_2\beta_2)(1-\Phi(x_1\beta_1))\right]$$

Bivariate Probit

- Is a joint model for two binary outcomes that generalizes the index function model from one latent variable to two latent variables that may be correlated
- Define the unobserved latent variables

$$y_1^* = \mathbf{x}_1'\beta_1 + \varepsilon_1$$
$$y_2^* = \mathbf{x}_2'\beta_2 + \varepsilon_2$$

  where the ε_1 and ε_2 are joint normal with means zero, variances one, and correlation ρ

- Then the **bivariate probit model** specifies the observed outcomes to be

$$y_1 = \begin{cases} 2 \text{ if } y_1^* > 0 \\ 1 \text{ if } y_1^* \leq 0 \end{cases}$$
$$y_2 = \begin{cases} 2 \text{ if } y_2^* > 0 \\ 1 \text{ if } y_2^* \leq 0 \end{cases}$$

Bivariate Probit

- The previous model collapses to two separate probit models for y_1 and y_2 when the error correlation $\rho = 0$
- For example,

$$\begin{aligned} p_{22} &= Pr[y_1 = 2, y_2 = 2] \\ &= Pr[y_1^* > 0, y_2^* > 0] \\ &= Pr[-\varepsilon_1 < \mathbf{x}_1'\beta_1, -\varepsilon_2 < \mathbf{x}_2'\beta_2] \\ &= Pr[\varepsilon_1 < \mathbf{x}_1'\beta_1, \varepsilon_2 < \mathbf{x}_2'\beta_2] \\ &= \int_{-\infty}^{\mathbf{x}_1'\beta_1} \int_{-\infty}^{\mathbf{x}_2'\beta_2} \phi(z_1, z_2, \rho) dz_1 dz_2 \\ &= \Phi(\mathbf{x}_1'\beta_1, \mathbf{x}_2'\beta_2, \rho) \end{aligned}$$

where $\phi(z_1, z_2, \rho)$ and $\Phi(z_1, z_2, \rho)$ are the standardized bivariate normal density and cdf for (z_1, z_2), respectively, with zero means, unit variances, and correlation ρ

Bivariate Probit

- Performing similar algebra for the other possible outcomes yields

$$\begin{aligned} p_{jk} &= Pr[y_1 = j, y_2 = k] \\ &= \Phi(q_1 \mathbf{x}_1' \beta_1, q_2 \mathbf{x}_2' \beta_2, \rho) \end{aligned}$$

where $q_l = 1$ if $y_l = 2$ and $q_l = -1$ if $y_l = 1$ for $l = 1, 2$.

- This is the basis for ML estimation

$$L = \sum_{i=1}^{N} \sum_{j=1}^{2} \sum_{k=1}^{2} y_{ijk} \ln p_{ijk}$$