Microeconometrics
Corner Solution and Censored Regression Models

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Examples of Censored Variables

- Censored
  - Evaluation of public programs to fight poverty (top-coding)
  - Unemployment duration
- Corner Solutions
  - Number of hours worked by married women
  - Private pension contributions
  - Expenditures of durable goods
  - Charity contributions

Censoring

With **censoring** we always observe the regressors $\mathbf{x}$, completely observe y^* for a subset of possible values of y^*, and incompletely observe y for the remaining possible values of y^*

- If censoring is **from below** (or from the left) we observe

$$y = \begin{cases} y^* \text{ if } y^* > L \\ L \text{ if } y^* \leq L \end{cases}$$

  Example: consumers may be sampled with positive expenditures in durable goods ($y^* > 0$) and others having zero expenditures ($y^* \leq 0$)

- If censoring is **from above** (or from the right) we observe

$$y = \begin{cases} y^* \text{ if } y^* < U \\ U \text{ if } y^* \geq U \end{cases}$$

  Example: top-coded annual income

Truncation

Truncation implies that all observations at the bound are lost.

With truncation **from below** we observe only

$$y = y^* \quad \text{if } y^* > L$$

Example: only consumers who purchased durable goods may be sampled (L=0)

With truncation **from above** we observe only

$$y = y^* \quad \text{if } y^* < U$$

Example: only low-income individuals may be sampled

Truncated Moments of the Standard Normal

Proposition

Suppose $z \sim \mathcal{N}[0,1]$. Then the left-truncated moments of z are

$$E[z|z>c] = \frac{\phi(c)}{1-\Phi(c)} \text{ and } E[z|z>-c] = \frac{\phi(c)}{\Phi(c)}$$

$$E[z^2|z>c] = 1 + \frac{c\phi(c)}{1-\Phi(c)}, \text{ and}$$

$$V[z|z>c] = 1 + \frac{c\phi(c)}{1-\Phi(c)} - \frac{\phi(c)^2}{[1-\Phi(c)]^2}$$

Truncated Moments of the Standard Normal

Suppose $u \sim \mathcal{N}[0, \sigma^2]$. Then

$$E[u|u > c] = \sigma \left[\frac{\phi(c/\sigma)}{1 - \Phi(c/\sigma)} \right]$$

Tobit Model

- The **censored normal regression model**, or **Tobit model**, is one with censoring from below at zero where the latent variable is linear in regressors wit additive error that is normally distributed and homoskedastic
- Thus

$$y^* = \mathbf{x}'\beta + \varepsilon$$

where $\varepsilon \sim \mathcal{N}[0, \sigma^2]$ has variance σ^2 constant across observations

- This implies that $y^* \sim \mathcal{N}[\mathbf{x}'\beta, \sigma^2]$
- y is defined as

$$y = \begin{cases} y^* \text{ if } y^* > 0 \\ - \text{ if } y^* \leq 0 \end{cases}$$

Tobit Model

- For *left-truncation* at zero we only observe y if $y^* > 0$. Then the left-truncated mean becomes

$$\begin{aligned} E[y] &= E[y^*|y^* > 0] \\ &= E[\mathbf{x}'\beta + \varepsilon|\mathbf{x}'\beta + \varepsilon > 0] \\ &= E[\mathbf{x}'\beta|\mathbf{x}'\beta + \varepsilon > 0] + E[\varepsilon|\mathbf{x}'\beta + \varepsilon > 0] \\ &= \mathbf{x}'\beta + E[\varepsilon|\mathbf{x}'\beta + \varepsilon > 0] \end{aligned}$$

assuming that ε is independent of $\mathbf{x}$

Tobit Model

- For *left-censored* at zero suppose we observe $y = 0$, and not only $y^* \leq 0$

$$\begin{aligned} E[y] &= Pr[d=0] \times E[y|d=0] + Pr[d=1] \times E[y|d=1] \\ &= 0 \times Pr[y^* \leq 0] + Pr[y^* > 0] \times E[y^*|y^* > 0] \\ &= Pr[y^* > 0] \times E[y^*|y^* > 0] \end{aligned}$$

where

- $Pr[y^* > 0] = 1 - Pr[y^* \leq 0] = Pr[\varepsilon > -\mathbf{x}'\beta]$
- $E[y^*|y^* > 0]$ is the truncated mean, and
- d is an indicator variable d=1 if $y > 0$ and d=0 if $y = 0$

Tobit Model

- Therefore the error term has truncated mean

$$E[\varepsilon|\varepsilon > -\mathbf{x}'\beta] = \sigma E\left[\frac{\varepsilon}{\sigma}\middle|\frac{\varepsilon}{\sigma} > -\frac{\mathbf{x}'\beta}{\sigma}\right] = \sigma\frac{\phi\left(-\frac{\mathbf{x}'\beta}{\sigma}\right)}{1-\Phi\left(-\frac{\mathbf{x}'\beta}{\sigma}\right)}$$
$$= \sigma\frac{\phi\left(\frac{\mathbf{x}'\beta}{\sigma}\right)}{\Phi\left(\frac{\mathbf{x}'\beta}{\sigma}\right)} = \sigma\lambda\left(\frac{\mathbf{x}'\beta}{\sigma}\right)$$

given symmetry about zero of $\phi(z)$ and where $\lambda(z) = \phi(z)/\Phi(z)$ is the **Inverse Mills Ratio**

Tobit Model

- Also,

$$Pr[\varepsilon > -\mathbf{x}'\beta] = Pr[-\varepsilon < \mathbf{x}'\beta] = Pr\left[-\frac{\varepsilon}{\sigma} < \frac{\mathbf{x}'\beta}{\sigma}\right] = \Phi\left(\frac{\mathbf{x}'\beta}{\sigma}\right)$$

- Then the conditional means given in the proposition specialize to
  - Latent variable: $E[y^*|\mathbf{x}] = \mathbf{x}'\beta$
  - Left-truncated (at 0): $E[y|\mathbf{x}, y > 0] = \mathbf{x}'\beta + \sigma\lambda\left(\frac{\mathbf{x}'\beta}{\sigma}\right)$
  - Left-censored (at 0): $E[y|\mathbf{x}] = \Phi\left(\frac{\mathbf{x}'\beta}{\sigma}\right)\mathbf{x}'\beta + \sigma\phi\left(\frac{\mathbf{x}'\beta}{\sigma}\right)$

Marginal Effects in the Tobit Model

Differentiating with respect to **x** yields

- Latent variable:

$$\frac{\partial E[y^*|\mathbf{x}]}{\partial \mathbf{x}} = \beta$$

- Left-truncated (at 0)

$$\begin{aligned}\frac{\partial E[y|\mathbf{x}, y > 0]}{\partial \mathbf{x}} &= \left\{1 - \frac{\mathbf{x}'\beta}{\sigma}\lambda\left(\frac{\mathbf{x}'\beta}{\sigma}\right) - \lambda\left(\frac{\mathbf{x}'\beta}{\sigma}\right)^2\right\}\beta \\ &= \left\{1 - \lambda\left(\frac{\mathbf{x}'\beta}{\sigma}\right)\left[\frac{\mathbf{x}'\beta}{\sigma} + \lambda\left(\frac{\mathbf{x}'\beta}{\sigma}\right)\right]\right\}\beta\end{aligned}$$

- Left-censored (at 0)

$$\frac{\partial E[y|\mathbf{x}]}{\partial \mathbf{x}} = \Phi\left(\frac{\mathbf{x}'\beta}{\sigma}\right)\beta$$

Maximum Likelihood Estimation

Consider ML estimation given censoring from below

- For $y > 0$ the density of y is the same as that for y^*, so $f(y|\mathbf{x}) = f^*(y|\mathbf{x})$ (where f(.) is the pdf)

- For $y = 0$ the density is discrete with mass equal to the probability of observing $y^* \leq 0$, $F^*(0|\mathbf{x})$ (where F(.) is the cdf)

Then the conditional density given censoring from below can be written as:

$$f(y|\mathbf{x}) = f^*(y|\mathbf{x})^d F^*(0|\mathbf{x})^{1-d}$$

where d is an indicator variable equal to 1 if $y > 0$ and equal to 0 if $y = 0$

Maximum Likelihood Estimation

Then the censored MLE maximizes

$$\ln L_N(\beta,\sigma) = \sum_{i=1}^{N} \mathbf{1}[y_i = 0] \ln\left[1 - \Phi\left(\frac{\mathbf{x}_i'\beta}{\sigma}\right)\right] + \\ + \mathbf{1}[y_i > 0]\left\{ \ln\left[\phi\left(\frac{y_i - \mathbf{x}_i'\beta}{\sigma}\right)\right] - \ln\sigma \right\}$$

given

$$\begin{aligned} F^*(0) &= Pr[y^* \leq 0] \\ &= Pr[\mathbf{x}'\beta + \varepsilon \leq 0] \\ &= \Phi(-\mathbf{x}'\beta/\sigma) \\ &= 1 - \Phi(\mathbf{x}'\beta/\sigma) \end{aligned}$$

where $\Phi(.)$ is the standard normal cdf and $f^*(y)$ is the $\mathcal{N}[\mathbf{x}'\beta, \sigma^2]$ density

R-squared

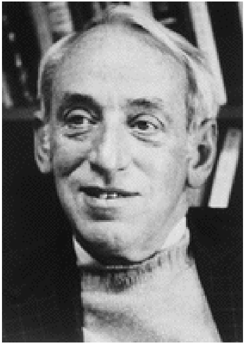
The coefficient of determination, R^2 can be written as the squared correlation coefficient between the actual y_i and the fitted values $\widehat{y}_i$

$$R^2 = [\rho(\widehat{y}_i, y_i)]^2$$

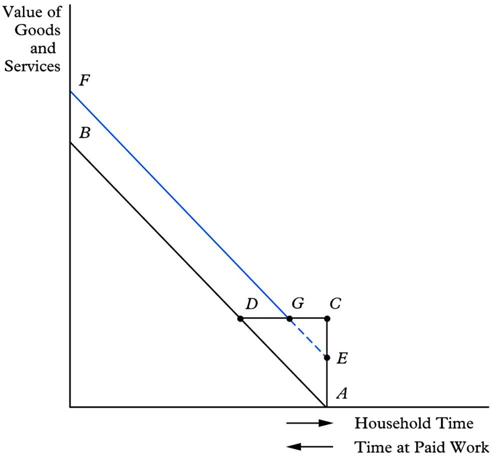
where

$$\widehat{y}_i = \Phi\left(\frac{\mathbf{x}_i'\widehat{\beta}}{\widehat{\sigma}}\right)\mathbf{x}_i'\widehat{\beta} + \widehat{\sigma}\phi\left(\frac{\mathbf{x}_i'\widehat{\beta}}{\widehat{\sigma}}\right)$$

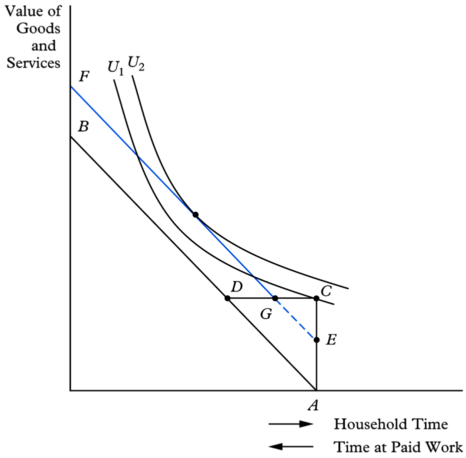
James Tobin (Nobel Prize 1981)



Budget constraints facing a single parent before and after child support assurance program adopted



A single parent who joins the labor force after child support assurance program adopted



Hurdle Model

More generally consider the latent model $y^* = \mathbf{x}'\beta + \varepsilon$ with variable censoring threshold $L = \mathbf{x}'\gamma$

This model is observationally equivalent to the latent model $y^* = \mathbf{x}'(\beta - \gamma) + \varepsilon$ with fixed threshold $L = 0$

Then

$$Pr[y = 0|\mathbf{x}] = 1 - \Phi(\mathbf{x}'\gamma)$$

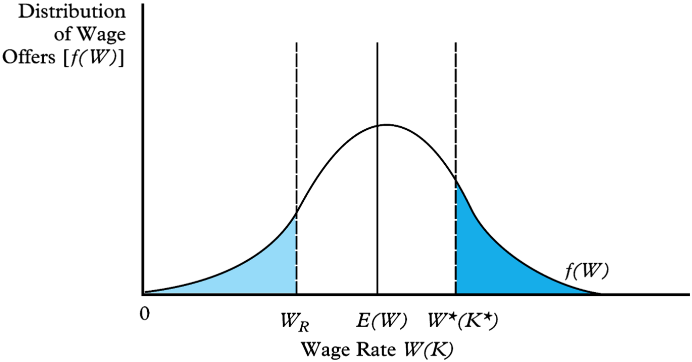
where

$$y|(\mathbf{x}, y > 0) \sim \mathcal{N}(\mathbf{x}'\beta, \sigma^2)$$

The log-likelihood function for the hurdle model is given by

$$\ln L_N(\beta, \sigma) = \sum_{i=1}^{N} \mathbf{1}[y_i = 0] \ln[1 - \Phi(\mathbf{x}_i'\gamma)] + \mathbf{1}[y_i > 0] \Big\{ \ln \Phi(\mathbf{x}_i'\gamma) + \\ + \ln \phi\left(\frac{y_i - \mathbf{x}_i'\beta}{\sigma}\right) - \ln(\sigma) \Big\}$$

Choice of Reservation Wage in a Model of Job Search



Truncated Regression Model

Consider the latent model

$$y^* = \mathbf{x}'\beta + \varepsilon$$

where $\varepsilon \sim \mathcal{N}[0, \sigma^2]$ has variance σ^2 constant across observations

Then

$$y = \mathbf{x}'\beta + \varepsilon \qquad \text{if } y^* > 0$$

For truncation from below at zero the conditional density of the observed y is

$$\begin{aligned} f(y) &= f^*(y|y > 0) \\ &= f^*(y)/Pr[y|y > 0] \\ &= f^*(y)/[1 - F^*(0)] \end{aligned}$$

Truncated MLE

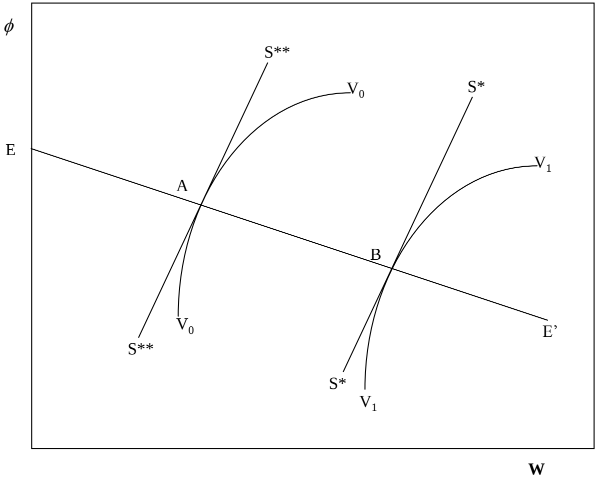
The truncated MLE maximizes

$$\ln L_N(\beta, \sigma^2) = \sum_{i=1}^{N} \left\{ \ln f^*(y_i|\mathbf{x}_i, \beta) - \ln[1 - F^*(0|\mathbf{x}_i, \beta)] \right\}$$

Therefore

$$\ln L_N(\beta, \sigma) = \sum_{i=1}^{N} \ln \phi\left(\frac{y_i - \mathbf{x}_i'\beta}{\sigma}\right) - \ln(\sigma) - \ln \Phi\left(\frac{\mathbf{x}_i'\beta}{\sigma}\right)$$

Equilibrium in the internal labor market



Example

Consider an empirical model defined by two equations which specifies both the probability of closure and wages as endogenous

Then, the reduced form of the equation system in the latent variables is:

$$\pi^*_{ijt} = \Pi_1 K_{ijt-1} + \varepsilon_{1ijt}, \qquad Y_{ijt} = 1(\pi^*_{ijt} < 0)$$

$$W^*_{ijt-1} = \Pi_2 K_{ijt-1} + \varepsilon_{2ijt-1}, \qquad W_{ijt-1} = \max(W_{Mit-1}, W^*_{ijt-1})$$

where K includes all exogenous variables in X, Z and U

$$\eta^c_{LL} = \bar{\phi}\left[\bar{W}\left(\frac{d\phi}{dw}\right)_{EE'}\right]^{-1}$$

where η^c_{LL} gives the percentage change in the probability that a plant closes in response to a 1% increase in the wage rate, holding product-market shocks constant

Estimated Elasticities

Quasi-elasticity of firm closure with respect to wages	0.460
Quasi-elasticity of firm closure with respect to minimum wage incidence	0.029
Elasticity of wages with respect to the probability of firm closure	-0.029