1. Consider the income fluctuation problem, in which a household chooses a consumption and asset plan $\{c_t(s^t), a_{t+1}(s^t)\}_{t\geq 0}$, given a stochastic endowment process $\{y_t(s^t)\}_{t\geq 0}$, to maximize:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t(s^t))$$

subject to

$$a_{t+1}(s^t) \geq 0, \qquad \forall s^t, t = 0, 1, ... \qquad (1)$$
$$c_t(s^t) + a_{t+1}(s^t) = (1+r)a_t(s^t) + y_t(s^t). \qquad \forall s^t, t = 0, 1, ... \qquad (2)$$

$\beta \in (0,1)$ is the discount factor; $c_t(s^t)$ is consumption at t, history s^t, a_t is asset holdings at time t, history s^t; $a_0(s_0)$ given. $a_{t+1}(s^t) \geq 0$ is a no-borrowing constraint. $y_t(s^t) > 0$ is non-financial income (wages, etc.); $(1+r) < 1/\beta$, is the constant gross return on savings. In each period, there is a realization of a stochastic event $s_t \in S$, and the history of events up to date t is what we denote by s^t. The unconditional probability of observing a particular sequence s^t is given by the probability $\pi_t(s^t)$. The household knows the true probability $\pi_t(s^t)$, hence $\mathbb{E}_0$ is the expectation operator taken with respect to $\pi_t(s^t)$.

   (a) Formulate the household consumption-savings problem, and obtain the first order conditions that describe its solution.

   (b) Show that an optimal consumption sequence satisfies the following conditions:

$$u'(c_t(s^t)) \geq \beta(1+r) \sum_{s^{t+1}|s^t} \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} u'(c_{t+1}(s^{t+1})), \quad \forall s^t, t.$$

   and

$$u'(c_t(s^t)) = \max\left\{\beta(1+r) \sum_{s^{t+1}|s^t} \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} u'(c_{t+1}(s^{t+1})), u'(w_t(s_t))\right\} \quad \forall s^t, t,$$

   with $w_t(s^t) = (1+r)a_t(s^t) + y_t(s^t)$.

2. Consider the Lucas tree economy, in which a household chooses consumption and asset shares $\{c_t(s^t), \pi_{t+1}(s^t)\}_{t\geq 0}$, given a stochastic dividend process

$\{y_t(s^t)\}_{t\geq 0}$, to maximize:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t(s^t))$$

subject to

$$0 \leq \pi_{t+1}(s^t) \leq 1, \qquad \forall s^t, t = 0, 1, ... \quad (3)$$

$$c_t(s^t) + p(s^t)\pi_{t+1}(s^t) = (y_t(s^t) + p_t(s^t))\pi(s^t). \qquad \forall s^t, t = 0, 1, ... \quad (4)$$

At the start of each period, the stochastic event s_t is realized, determining the dividend payment in the same period. s_t can take two values: $y(s_t = s_L) = y_L$ and $y(s_t = s_H) = y_H$.

(a) Suppose that dividends follow an i.i.d. process, with $\text{Prob}(s_t = s_L) = p$ and $\text{Prob}(s_t = s_H) = 1 - p$. Formulate the consumer's recursive optimization problem. Determine the equilibrium price-dividend ratio, $p(y_i)/y_i$, $i = H, L$, when dividends follow the i.i.d. process described above.

(a) Suppose that dividends follow a Markov process, with $\text{Prob}(s_{t+1} = s_L|s_t = s_L) = p$, $\text{Prob}(s_{t+1} = s_H|s_t = s_H) = q$, $\sum_{i=L,H} \text{Prob}(s_{t+1} = s_i|s_L) = \sum_{i=L,H} \text{Prob}(s_{t+1} = s_i|s_H) = 1$. Formulate the consumer's recursive optimization problem. Determine the equilibrium price-dividend ratio, $p(y_i)/y_i$, $i = H, L$, when dividends follow the Markov process described above.

3. Consider a Bewley economy as in Huggett (1993), as discussed in the lecture slides and in Section 18.2.3 in L&S (2018). Using the calibration of the model as in Huggett, solve numerically for the competitive equilibrium using the algorithm described in the paper.

(a) Plot the optimal policy functions, $a'(a, e_1)$ and $a'(a, e_2)$. Represent in a plot and explain what is the impact the credit limit $\underline{a} \in \{-2, -4, -6, -8\}$ on the policy rules.

(b) For a given interest rate r, compute average assets in the economy $\mathbb{E}a(r)$. Plot the function $\mathbb{E}a(r)$ on a graph with assets on the horizontal axis and the interest rate on the vertical axis. As in a., show what is the impact of the credit limit on $\mathbb{E}a(r)$.

(c) Plot the stationary distributions of the two economies $\underline{a} \in \{-4, -8\}$, $\bar{\lambda}(a) = \sum_{e=e_1,e_2} \pi(e)\lambda(a, e)$, where $\pi(e)$ is the unconditional distribution over the employment states e and $\lambda(a, e)$ is the joint distribution over (a, e) pairs. What is the impact of the credit limit on the mean and variance of assets in the cross-section of the economy?

References:

Ljunqgvist, L. and Sargent T. J. (2018), "Recursive Macroeconomic Theory", fourth edition, The MIT Press.

Hugget, M. (1993), *'The risk-free rate in heterogeneous-agent incomplete-insurance economies'*.