1. Endogenous Government Expenditure. Consider the following economy populated by a government, a representative household, and two representative firms. The government sets both taxes and a sequence of government expenditures $\{g_t\}_{t=0}^{\infty}$. The preferences of the household are ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where u is the utility function, $\beta \in (0,1)$, c_t is period t consumption. The budget constraint is:

$$\sum_{t=0}^{\infty} p_t(c_t + x_t) \leq \sum_{t=0}^{\infty} p_t(1-\tau_{kt})r_t k_t, \tag{2}$$

where p_t is the price of period t goods in units of a numeraire, x_t is investment, τ_{kt} is the tax rate on capital income, and r_t the rental price of capital, k_t.

In this economy, g_t is a publicly provided private good that has a direct impact on the effectiveness of investment. Specifically, the capital stock evolves according to the following technology:

$$k_{t+1} \leq G(x_t, g_t) + (1-\delta)k_t, \quad t = 0, 1, ... \tag{3}$$

with k_0 given, $\delta \in (0,1)$, and where we assume that G is homogeneous of degree one: $G(x_t, g_t) = Bx_t^{\gamma} g_t^{1-\gamma}$, $B > 0$ and $0 < \gamma < 1$. The consumer takes g_t takes given when deciding on private investment x_t.

Production is linear in capital:

$$f(k_t) = Ak_t, \quad t = 0, 1, ... \tag{4}$$

with $A > 0$. The resource constraints are:

$$c_t + g_t + x_t = f(k_t), \quad t = 0, 1, ... \tag{5}$$

a. (3 pt) For a given budget-feasible sequence of government policies $\{g_t, \tau_{kt}\}_{t=0}^{\infty}$, define a tax-distorted competitive equilibrium, formulate the household's and the firm's problem, and derive the competitive equilibrium conditions.

A competitive equilibrium with taxes is a budget-feasible government policy, a feasible allocation, and prices such that, given the prices and the government policy, the allocation solves the household's problem and the firm's problem. [0.5 pt]

The household's problem is to choose $\{c_t, x_t, k_{t+1}\}_{t=0}^{\infty}$ *to maximize:*

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to (2) *and* (3), *with* k_0 *given. [0.25 pt]*

The firm's problem is to choose k_t *every period to maximize:*

$$f(k_t) - r_t k_t.$$

[0.25 pt]

The competitive equilibrium conditions from the household's problem can be obtained by maximizing the Lagrangian with multipliers φ *and* λ_t*:*

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \varphi \left[\sum_{t=0}^{\infty} p_t (1-\tau_{kt}) r_t k_t - \sum_{t=0}^{\infty} p_t (c_t + x_t) \right] + \sum_{t=0}^{\infty} \lambda_t \left[G(x_t, g_t) + (1-\delta) k_t - k_{t+1} \right]$$

First order conditions:

$$\begin{aligned}
c_t: &\quad \beta^t u_c(t) - \varphi p_t &= 0, \quad (6)\\
x_t: &\quad -\varphi p_t + \lambda_t G_x(t) &= 0, \quad (7)\\
k_{t+1}: &\quad -\lambda_t + \varphi p_{t+1}(1-\tau_{kt+1}) r_{t+1} + \lambda_{t+1}(1-\delta) &= 0. \quad (8)
\end{aligned}$$

together with the constraints (2), *and* (3) *for* $t = 0, 1, ...$ *and* k_0 *given. These can be summarized by:*

$$\frac{u_c(t)}{\beta u_c(t+1)} = \frac{G_x(t)}{G_x(t+1)} \left[1 - \delta + (1-\tau_{kt+1}) r_{t+1} G_x(t+1) \right], \tag{9}$$

together with (2) *and* (3) *for* $t = 0, 1, ...$ *and* k_0 *given.*

The equilibrium condition from the firm's problem is:

$$r_t = f_k(t), \ t = 0, 1, ... \tag{10}$$

The competitive equilibrium conditions are given by (2), (3), (9), (10), *and* (5) *(the government budget constraint is implied by* (2) *and* (5)*). [2 pt]*

b. (4 pt) The consumer's preferences are given by: $u(c_t) = c_t^{1-\sigma}/(1-\sigma), \sigma > 0$. Using the equilibrium conditions, show that with a constant capital income tax, $\tau_{kt} = \tau$, and a constant ratio of government investment to capital $g_t/k_t = \theta_g$, the equilibrium is consistent with a balanced path in which $c_{t+1}/c_t = k_{t+1}/k_t = \gamma$. Provide conditions that determine $\gamma, x_t/k_t$, and c_t/k_t along the equilibrium path. Describe the transition between period 0, when $k_t = k_0$, and the constant growth path with $k_{t+1}/k_t = \gamma$.

With the utility function given above, (9) *becomes:*

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \frac{\beta G_x(t)}{G_x(t+1)}\left[1-\delta+(1-\tau_{kt+1})r_{t+1}G_x(t+1)\right] \tag{11}$$

Replacing in r_t from the equilibrium conditions and $G_x(t) = \gamma B(x_t/g_t)^{\gamma-1}$:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \frac{\beta(x_t/g_t)^{\gamma-1}}{(x_{t+1}/g_{t+1})^{\gamma-1}}\left[1-\delta+(1-\tau_{kt+1})A\gamma B\left(\frac{x_{t+1}}{g_{t+1}}\right)^{\gamma-1}\right] \tag{12}$$

Under the constant policy assumption, this becomes:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \frac{\beta(x_t/k_t)^{\gamma-1}}{(x_{t+1}/k_{t+1})^{\gamma-1}}\left[1-\delta+(1-\tau)A\gamma B\left(\frac{x_{t+1}}{\theta_g k_{t+1}}\right)^{\gamma-1}\right] \tag{13}$$

[1.5 pt]

From the resources constraint:

$$\frac{c_t}{k_t} + \frac{x_t}{k_t} + \theta_g = A \tag{14}$$

And the capital accumulation equation:

$$\frac{k_{t+1}}{k_t} = B\left(\frac{x_t}{k_t}\right)^{\gamma}\theta_g^{1-\gamma} + 1 - \delta \tag{15}$$

[1 pt]

Along a balanced path with $c_{t+1}/c_t = k_{t+1}/k_t = \mu$, $c_t/k_t = c/k$, and $x_t/k_t = x/k$ (i.e. constant ratios), the following three (non-linear) conditions jointly determine $\mu, c/k, x/k$:

$$\mu^\sigma = \beta\left[1-\delta+(1-\tau)A\gamma B\left(\frac{x}{k}\right)^{\gamma-1}\theta_g^{1-\gamma}\right] \tag{16}$$

$$A = \frac{c}{k}+\frac{x}{k}+\theta_g \tag{17}$$

$$\mu = B\left(\frac{x}{k}\right)^{\gamma}+1-\delta \tag{18}$$

[1 pt]

With $g_t/k_t = \theta_g$, in particular with $g_0 = \theta_g k_0$, the economy starts from the balanced path described above in period 0 and there is no transition period (i.e. there is constant growth from $t = 0$). [0.5 pt]

c. (2 pt) Consider now the problem of a government planner who decides on sequences $\{g_t, \tau_{kt}\}_{t=0}^\infty$ to maximize the representative consumer utility, subject to the constraints imposed by a competitive equilibrium. Show that, using the competitive equilibrium conditions, the consumer's present value budget constraint (2) can be recast in terms of the allocation, as:

$$\sum_{t=0}^\infty \beta^t\left[u_c(t)c_t - u_c(t)g_t\frac{G_g(t)}{G_x(t)}\right] = u_c(0)W_0, \tag{19}$$

where $W_0 = [(1-\tau_{k0})f_k(0)]k_0$, $u_c(t)$ denotes marginal utility of consumption in period t, and $G_x(t), G_g(t)$ the first derivatives of G with respect to x_t and g_t, respectively.

Since G is linearly homogeneous:

$$G(x_t, g_t) = G_x(t)x_t + G_g(t)g_t. \tag{20}$$

The consumer's budget constraint (2) can be written (with equality) as:

$$\sum_{t=0}^\infty p_t\left(c_t + \frac{G(x_t,g_t)-g_tG_g(t)}{G_x(t)}\right) = \sum_{t=0}^\infty p_t(1-\tau_{kt})r_tk_t, \tag{21}$$

which in turn, since $G(x_t, g_t) = k_{t+1}-(1-\delta)k_t$, is equivalent to:

$$\sum_{t=0}^\infty p_t\left(c_t - g_t\frac{G_g(t)}{G_x(t)}\right) = \sum_{t=0}^\infty p_t(1-\tau_{kt})r_tk_t - \sum_{t=0}^\infty p_t\left(\frac{k_{t+1}-(1-\delta)k_t}{G_x(t)}\right). \tag{22}$$

It may be informative to rearrange terms:

$$\sum_{t=0}^{\infty} p_t\left(c_t - g_t\frac{G_g(t)}{G_x(t)}\right) = \sum_{t=0}^{\infty} p_t\left((1-\tau_{kt})r_t + \frac{1-\delta}{G_x(t)}\right)k_t - \sum_{t=0}^{\infty} p_t\frac{k_{t+1}}{G_x(t)}. \qquad (23)$$

Notice that the k_t terms is value of the capital stock in all periods. The difference between that value and the capitalized value of capital from period 1 onward (the last term in the previous equation) is the value of initial capital:

$$\sum_{t=0}^{\infty} p_t\left((1-\tau_{kt})r_t + \frac{1-\delta}{G_x(t)}\right)k_t - \sum_{t=0}^{\infty} p_t\frac{k_{t+1}}{G_x(t)} = p_0\left[(1-\tau_{k0})r_0 + \frac{1-\delta}{G_x(0)}\right]k_0 \qquad (24)$$

To see this, note that:

$$\begin{aligned}
&\sum_{t=0}^{\infty} p_t\left[(1-\tau_{kt})r_t + \frac{1-\delta}{G_x(t)}\right]k_t - \sum_{t=0}^{\infty} p_t\frac{k_{t+1}}{G_x(t)} = \\
&= p_0\left[(1-\tau_{k0})r_0 + \frac{1-\delta}{G_x(0)}\right]k_0 + \\
&+ \sum_{t=0}^{\infty} p_{t+1}\left[(1-\tau_{kt+1})r_{t+1} + \frac{1-\delta}{G_x(t+1)}\right]k_{t+1} - \sum_{t=0}^{\infty} p_t\frac{k_{t+1}}{G_x(t)} \\
&= p_0\left[(1-\tau_{k0})r_0 + \frac{1-\delta}{G_x(0)}\right]k_0 + \sum_{t=0}^{\infty} p_t\frac{k_{t+1}}{G_x(t)} - \sum_{t=0}^{\infty} p_t\frac{k_{t+1}}{G_x(t)} \\
&= p_0\left[(1-\tau_{k0})r_0 + \frac{1-\delta}{G_x(0)}\right]k_0 \qquad (25)
\end{aligned}$$

where we have used the Euler condition (9) and

$$\frac{p_t}{p_0} = \beta^t\frac{u_c(t)}{u_c(0)}, \qquad (26)$$

to use the equilibrium condition:

$$\frac{p_{t+1}}{G_x(t+1)}\left[(1-\tau_{kt+1})r_{t+1}G_x(t+1) + 1 - \delta\right] = \frac{p_t}{G_x(t)}. \qquad (27)$$

We conclude that the budget constraint can be written as:

$$\sum_{t=0}^{\infty} p_t\left[c_t - g_t\frac{G_g(t)}{G_x(t)}\right] = p_0\left[(1-\tau_{k0})r_0 + \frac{1-\delta}{G_x(0)}\right]k_0. \qquad (28)$$

Replace prices to obtain the implementability condition:

$$\sum_{t=0}^{\infty} \beta^t \left[u_c(t) c_t - u_c(t) g_t \frac{G_g(t)}{G_x(t)} \right] = u_c(0) \left[(1 - \tau_{k0}) f_k(0) + \frac{1-\delta}{G_x(0)} \right] k_0. \tag{29}$$

[2 pt]

d. (2 pt) Formulate the Ramsey problem, where the planner is choosing time paths for c_t, k_{t+1}, x_t, g_t, and obtain the first order conditions that describe the solution.

The Ramsey problem is to maximize lifetime utility (1) subject to the implementability condition (19) and the resource constraints (3) and (5). [1 pt]

Let φ be the Lagrange multiplier on the implementability condition and define:

$$V(c_t, g_t, x_t, \varphi) \equiv u(c_t) + \varphi \left[u_c(t) c_t - u_c(t) g_t \frac{G_g(t)}{G_x(t)} \right]$$

The Lagrangian that corresponds to the Ramsey problem is:

$$J = \sum_{t=0}^{\infty} \beta^t [V(c_t, g_t, x_t, \varphi) + \theta_t^1 (f(k_t) - c_t - g_t - x_t) + \theta_t^2 (G(x_t, g_t) + (1-\delta) k_t - k_{t+1}] - \varphi \mathcal{W}_0$$

[0.5 pt]
First order conditions:

$$\begin{aligned}
c_t: & \quad V_c(t) - \theta_t^1 & = 0, \quad t \geq 0 \quad (30)\\
g_t: & \quad V_g(t) - \theta_t^1 + \theta_t^2 G_g(t) & = 0, \quad t \geq 0 \quad (31)\\
x_t: & \quad V_x(t) - \theta_t^1 + \theta_t^2 G_x(t) & = 0, \quad t \geq 0 \quad (32)\\
k_{t+1}: & \quad -\theta_t^2 + \beta[\theta_{t+1}^1 f_k(k_{t+1}) + \theta_{t+1}^2 (1-\delta)] & = 0, \quad t \geq 0, \quad (33)
\end{aligned}$$

together with the constraints. [0.5 pt]
The marginal conditions can be summarized by:

$$\frac{V_c(t) - V_x(t)}{V_c(t+1) - V_x(t+1)} = \beta \frac{G_x(t)}{G_x(t+1)} \left[\frac{V_c(t+1)}{V_c(t+1) - V_x(t+1)} f_k(k_{t+1}) G_x(t+1) + 1 - \delta \right], \tag{34}$$

$$\frac{V_c(t) - V_g(t)}{G_g(t)} = \frac{V_c(t) - V_x(t)}{G_x(t)}. \tag{35}$$

e. (2 pt) Assume that the planner takes $\mathcal{W}_0 \equiv u_c(0)W_0$ as given, and that there is a value for τ_{k0} that satisfies the implementability condition (19), evaluated at the solution. Compare the solution to the Ramsey problem and the competitive equilibrium conditions to determine the optimal sequence of capital income tax, τ_{kt}.

The relevant condition is (34). Firstly, we have that given the given functional forms[1]*:*

$$V_c(t) = u_c(t)\left\{1 + \varphi\left[1 - \sigma\left(1 - \gamma^r\left(\frac{x_t}{c_t}\right)\right)\right]\right\}, \tag{36}$$

$$V_x(t) = -\varphi\gamma^r u_c(t), \tag{37}$$

with $\gamma^r = (1-\gamma)/\gamma$*. [1 pt]*
On the right hand side of (34), the term

$$\frac{V_c(t+1)}{V_c(t+1) - V_x(t+1)} f_k(k_{t+1})G_x(t{+}1) = \frac{1}{1 - V_x(t+1)/V_c(t+1)} f_k(k_{t+1})G_x(t{+}1) \tag{38}$$

implies that $\tau_{kt+1} = 0$ *only if* $V_x(t) = 0$*, which could only hold if* $\varphi = 0$*. That is, provided there is need to collect tax revenue beyond* $t = 0$ *(i.e.* $\varphi > 0$*) it is optimal to set a non-zero capital income tax in every period. [1 pt]*

2. A Simple Lucas Tree There is a single asset in the economy that pays dividends y_t every period. Dividends can be high or low: $y_t \in \{y_L, y_H\}$, and follow a Markov process with $\text{Prob}(y_{t+1} = y_L | y_t = y_L) = p$, $\text{Prob}(y_{t+1} = y_H | y_t = y_H) = q$, $\sum_{i=L,H} \text{Prob}(y_{t+1} = y_i | y_L) = \sum_{i=L,H} \text{Prob}(y_{t+1} = y_i | y_H) = 1$. A representative consumer chooses consumption c_t and the share of asset holdings, $\pi_t \in [0,1]$, with preferences given by:

$$\sum_{t=0}^{\infty} \frac{c_t^{1-\sigma}}{1-\sigma}, \ \sigma > 0. \tag{39}$$

a. (2 pt) The consumer takes the price of the asset $p(y)$ as given. Formulate the consumer's recursive optimization problem, with a Bellman equation and budget constraint.

The Bellman equation for the consumer's problem is:

$$v(y,\pi) = \max_{c,\pi'}\{u(c) + \beta \sum_{i=L,H} Prob(y_i|y)v(y_i,\pi')\}, \tag{40}$$

[1] Note that if $\gamma = 1$, we have the standard model in which investment depends on private investment alone, and the expression is the same as in the standard Ramsey problem studied in class.

subject to:

$$c + p(y)\pi' = (p(y) + y)\pi, \tag{41}$$

$$\pi' \in [0, 1]. \tag{42}$$

[2 pt]

b. (1 pt) The equilibrium price is such that the consumer is optimizing with $c_t = y_t$ and $\pi_t = 1$ in every period. Derive the asset price when dividends follow the Markov process described above.

The first order conditions of the consumer's problem together with market clearing imply the following equation:

$$p(y)u_c(y) = \beta \sum_{i=L,H} u_c(y_i)Prob(y_i|y)(p(y_i) + y_i), \quad y = y_L, y_H. \tag{43}$$

[0.5 pt]
Given that $Prob(y_i = y_L|y = y_L) = p$, *and* $Prob(y_i = y_H|y = y_H) = q$, *we can write this condition as:*

$$\begin{bmatrix} p(y_L) \\ p(y_H) \end{bmatrix} \begin{bmatrix} u_c(y_L) & 0 \\ 0 & u_c(y_H) \end{bmatrix} = \beta \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix} \begin{bmatrix} u_c(y_L) & u_c(y_L) \\ u_c(y_H) & u_c(y_H) \end{bmatrix} \left(\begin{bmatrix} p(y_L) \\ p(y_h) \end{bmatrix} + \begin{bmatrix} y_l \\ y_h \end{bmatrix} \right)$$

Define:

$$\mathbf{p} \equiv \begin{bmatrix} p(y_L) \\ p(y_L) \end{bmatrix}; \mathbf{u_c} \equiv \begin{bmatrix} u_c(y_L) & 0 \\ 0 & u_c(y_H) \end{bmatrix}; \mathbf{yu_c} \equiv \begin{bmatrix} y_L u_c(y_L) \\ y_H u_c(y_H) \end{bmatrix}; \mathbf{P} \equiv \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}.$$

The asset price is given by:

$$\mathbf{p} = \mathbf{u_c}^{-1}(\mathbf{I} - \beta\mathbf{P})^{-1}\beta\mathbf{P}\mathbf{yu_c}. \tag{44}$$

[0.5 pt]

3. A career ladder Each period a previously unemployed worker draws one offer to work at a non-negative wage w, where w is governed by a cumulative distribution function F that satisfies $F(0) = 0$ and $F(B) = 1$ for some $B > 0$. The worker seeks to maximize the expected value of lifetime earnings:

$$\sum_{t=0}^{\infty} \beta^t y_t$$

where $\beta \in (0,1)$ and $y_t = w$ if the worker is employed with wage w and $y_t = b$ if the worker is unemployed. At the beginning of each period a worker employed at wage w the previous period has the probability $\alpha \in (0,1)$ of getting a promotion, which means that he will earn a wage γw with $\gamma > 1$. This wage will prevail until a next promotion.

a. (2 pt) Formulate a Bellman equation for a previously employed worker.

The value of a job with wage w, $v(w)$, is given by:

$$v(w) = w + \beta[\alpha \cdot v(\gamma w) + (1-\alpha) \cdot v(w)]. \tag{45}$$

[1.5 pt]
This implies a linear function in w:

$$v(w) = \kappa w,\ \kappa = [1 - \beta(\alpha\gamma + 1 - \alpha)]^{-1}. \tag{46}$$

[0.5 pt]

b. (1 pt) Formulate a Bellman equation for a previously unemployed worker.

The Bellman equation of an unemployed worker is:

$$v_u(w) = \max\left\{v(w), b + \beta \int_0^B v_u(w)dF\right\} \tag{47}$$

c. (1 pt) Describe the decision rule (accept or reject offer) for an unemployed worker.

The worker accepts an offer provided that $w \geq \bar{w}$, for some reservation wage $\bar{w}$. The reservation wage is defined by:

$$v(\bar{w}) = b + \beta\bar{v}_u, \quad \bar{v}_u = \int_0^B v_u(w)dF. \tag{48}$$

Therefore the reservation wage solves:

$$\kappa\bar{w} = b + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta} dF + \beta \int_{\bar{w}}^B \kappa w dF \tag{49}$$

Total: 20 points.