Macroeconomics II

– Preliminary –

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João Brogueira de Sousa

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NOVA SCHOOL OF BUSINESS & ECONOMICS

Optimal Savings Problems

- A consumer wants to maximize the expected discounted sum of utility of one-period consumption flows.
- In contrast to previous lectures, he is cut off from all insurance and almost all asset markets.
- There is a single risk-free asset.
- The consumer manages variations over time on his risk-free asset holdings to "self-insure" fluctuations in consumption.

Environment

- A household chooses a state-contingent consumption plan $\{c_t\}_{t\geq 0}$ to maximize:

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_t)$$

subject to

$$\begin{aligned} 0 \leq c_t &\leq a_t, & t = 0, 1, ... \quad (1) \\ a_{t+1} &= R(a_t - c_t) + Y_{t+1}. & t = 0, 1, ... \quad (2) \end{aligned}$$

- $\beta \in (0, 1)$ is the discount factor;
- c_t is consumption at t, a_t is asset holdings at time t;
- $c_t \leq a_t$ is the borrowing constraint;
- Y_t is non-financial income (wages, etc.);
- $R = 1 + r$, where $r > 0$ is the constant interest rate on savings.

Some Questions

- How can we describe an optimal consumption plan?
- Under which conditions does it exist? And is it unique?
- If so, given $\{Y_t\}_{t>0}$, what does it imply for $\{c_t, a_{t+1}\}$ for $t = 0, 1, ...$?
- Is the model useful to generate time series of $\{c_t, a_{t+1}\}$ we can compare to the data?
- And does it have predictions about the cross-section distribution of (c, a, Y)?
- If so, we have a theory that relates income, consumption and wealth at the household level (and as we'll see later, at the aggregate level too).
- Quite useful to think about: taxation, redistribution policies, stabilization policies, etc.
- The "common ancestor" (L&S (2018), p.3) of modern macroeconomic theory.

Environment

Timing

The timing of events is as follows:

1. At the start of period t, the household chooses consumption c_t.
2. Labor is supplied during the period, and income Y_{t+1} is received at the end of the period t.
3. Financial income $R(a_t - c_t)$ is received at the end of period t.
4. The process repeats in $t + 1$.

Environment

Income

- Non-financial income follows an exogenous process.

$$Y_t = y(Z_t), \tag{3}$$

  where Z_t is an exogenous process.

- We assume that Z_t follows a finite state Markov chain, taking values $\mathbf{Z}$ with transition matrix P.

Environment

Assumptions

We assume that:

- $\beta R < 1$ (more on this assumption later).
- u is strictly increasing, strictly concave, smooth function with:

$$\lim_{c \to 0} u'(c) = \infty, \qquad \lim_{c \to \infty} u'(c) = 0. \tag{4}$$

The asset space is $\mathbb{R}^+$ and the state $(a, z) \in \mathbf{S} = \mathbb{R}^+ \times \mathbf{Z}$.

Feasible Plans

A feasible consumption plan from state $(a, z) \in \mathbf{S}$ is a consumption sequence $\{c_t\}$ and asset holdings $\{a_t\}$ that satisfy:

- $(a_0, z_0) = (a, z)$;
- conditions (1) and (2);
- measurability: c_t can be chosen as a function of events up to date t, but not after.

Value Function

A value function $V : \mathbf{S} \rightarrow \mathbb{R}$ is defined by

$$V(a, z) = \max \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{5}$$

s.t. feasibility of $\{c_t\}$ from the initial state (a, z).

Euler Equation with Borrowing Constraints

The Euler equation that characterizes optimal consumption plans is:

$$u'(c_t) \geq \beta R \mathbb{E}_t u'(c_{t+1}). \tag{6}$$

Specifically:

$$u'(c_t) = \begin{cases} \beta R \mathbb{E}_t u'(c_{t+1}), \text{ if } c_t < a_t, \\ \beta R \mathbb{E}_t u'(c_{t+1}) + \theta_t(a_t, z_t), \text{ if } c_t = a_t. \end{cases} \tag{7}$$

where $\theta_t > 0$ is a multiplier on the constraints $c_t \leq a_t$.
At the borrowing constraint, $c_t = a_t$. Therefore (7) is equivalent to:

$$u'(c_t) = \max\{\beta R \mathbb{E}_t u'(c_{t+1}), u'(a_t)\} \tag{8}$$

Optimality

Ma et al. (2020) provide conditions such that:

1. For each $(a, z) \in \mathbf{S}$, there exists a *unique* optimal consumption path from (a, z);
2. This unique feasible consumption path from (a, z) satisfies the Euler condition (8) and the transversality condition:

$$\lim_{t \to \infty} \beta^t \mathbb{E}[u'(c_t) a_{t+1}] = 0. \tag{9}$$

In addition, there exists a unique optimal consumption function $\sigma^* : \mathbf{S} \to \mathbb{R}^+$ such that the path from (a, z) generated by:

$$(a_0, z_0) = (a, z), \quad c_t = \sigma^*(a_t, Z_t), \quad a_{a+1} = R(a_t - c_t) + Y_{t+1},$$

satisfies (8) and (9).
The solution to the savings problem is the policy function σ^*.

Optimality

Existence and Uniqueness Conditions

Ma et al. (2020) provide conditions for the more general case with, in addition to random Y_t:

- Stochastic returns: R_t
- Time varying and possibly random discount rates: β_t

These are conditions of the sort:

- Bound lim $\left(\Pi\beta_t\right) < 1$, equivalent to $\beta < 1$ with constant β.
- Bound lim $\left(\Pi\beta_t R_t\right) < 1$, equivalent to $\beta R < 1$ with constant β, R.

Additionally, if Y has a finite first moment and $\mathbb{E}u'(Y) < \infty$:

$\Longrightarrow$ Any feasible consumption plan that satisfies the Euler and the transversality conditions is an optimal policy, such plan exists and is unique, and can be found by iterating on a functional Euler equation.

Solution Methods

Value Function Iteration

One could write a standard Bellman Equation:

$$v(a, z) = \max_{c,\hat{a}} \left\{ u(c) + \beta \mathbb{E} v(\hat{a}, \hat{z}) \right\}$$

subject to:

$$\hat{a} = R(a - c) + Y(\hat{z})$$
$$0 \leq c \leq a$$

and solve it using value function iteration.

Instead, here we solve for the optimal policy σ^* directly, using the Euler condition as a functional equation:

$$(u' \circ \sigma)(a, z) =$$
$$= \max \left\{ \beta R \mathbb{E}_z[(u' \circ \sigma)(R(a - \sigma(a, z)) + Y(\hat{z}), \hat{z})], u'(a) \right\}$$

- $(u' \circ \sigma)(s) = u'(\sigma(s))$.
- $\mathbb{E}_z$ is the expectation operator conditioning on state z information.
- $\hat{z}$ is next period random variable z.
- The unknown is function σ.

How is this useful?

Functional Operator

Let $\mathscr{C}$ be the space of continuous functions $\sigma : \mathbf{S} \to \mathbb{R}$ such that:

- σ is increasing in the first argument;
- $0 < \sigma(a, z) \leq a$ for all $(a, z) \in \mathbf{S}$; and

$$\sup_{(a,z)\in\mathbf{S}} \left|(u' \circ \sigma)(a, z) - u'(a)\right| < \infty.$$

Let $K : \mathscr{C} \to \mathscr{C}$ be a functional operator on $\mathscr{C}$ such that:

$$(K\sigma)(a, z) = \Bigg\{ c \in [0, a] :$$

$$u'(c) = \max_{c} \left\{ \beta R \mathbb{E}_z[(u' \circ \sigma)(R(a - c) + Y(\hat{z}), \hat{z})], u'(a) \right\} \Bigg\}$$

Fixed points of K coincide with solutions to the functional equation (i.e. the optimal policy):

$$\sigma^* = \sigma \in \mathscr{C} : (K\sigma)(a, z) = \sigma, \ \forall (a, z) \in \mathbf{S}.$$

Functional Operator

Solution

Policy Function Iteration (Coleman (1990)):

Ma et al. (2020) show that the unique optimal policy σ^* can be computed by picking any initial guess $\sigma_0 \in \mathscr{C}$ and iterating with the operator K:

1. Pick an initial guess $\sigma_0 \in \mathscr{C}$;
2. Apply the operator $(K\sigma_0)(a, z)$ for all $(a, z) \in \mathbf{S}$ to obtain $\sigma_1 = K\sigma_0$.
3. If $\sigma_1 = \sigma_0$, then σ_0 is the optimal policy. If not, update guess to σ_1 and apply K: $\sigma_2 = K\sigma_1$.
4. Proceed in this fashion until convergence: $\sigma^* = K\sigma^*$.

Wealth Distribution

The optimal policy σ^* and the Markov transition $\mathbf{P}$ induce a distribution over states $(a, z) \in S$ via the law of motion of a_{t+1}, condition (2):

$$a_{t+1} = R\big[a_t - \sigma^*(a_t, Z_t)\big] + Y(Z_{t+1}) \tag{10}$$

$$Z_{t+1} \sim P(Z_t, :) \tag{11}$$

Let Q be the joint stochastic kernel of (a_t, Z_t). Q tells the probabilities of moving from state s to other states $\hat{s}$ in one time step.

Under some technical conditions on $\beta, R, u'(c)$, we obtain existence, uniqueness and stationarity of Q on $\mathbf{S}$.

This is required to establish existence of stationary equilibria in heterogeneous agent models with risk and incomplete markets (Huggett (1993), Aiyagari (1994)).

Diverging Wealth with $\beta R = 1$

Suppose instead that $\beta R = 1$.

The Euler condition with an occasionally binding constraint is:

$$u'(c_t) \geq \beta R \mathbb{E}\left[u'(c_{t+1})\right] \iff u'(c_t) \geq \mathbb{E}\left[u'(c_{t+1})\right] \qquad (12)$$

$u'(c_t)$ is a nonnegative supermartingale: converges almost surely.

It is possible to show that it converges to zero: $c_t \rightarrow_{as} \infty$.

We need to impose $\beta R < 1$ so that c and a are bounded. Or to write a model of R as an equilibrium outcome such that $R < \beta^{-1}$.

Readings

- L&S (2018): 17
- Ma et al. (2030): *"The income fluctuation problem and the evolution of wealth"*.
- Numerical examples: quantecon.org lecture.