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1. Consider an exchange economy with I consumers, denoted by $i = 1, ..., I$. Each consumer i owns a stochastic endowment of goods each period, that depends on the history s^t up to that period, $y_t^i(s^t)$. The consumer purchases a history-dependent consumption plan, $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$, and orders these plans according to:

$$U_i(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) u_i(c_t^i(s^t)),\ 0 < \beta < 1.$$

Assume that $u_i(c) = \log(c)$ for all $i = 1, ..., I$, and $c > 0$. Show that in a Pareto Optimal (feasible) allocation, individual consumption satisfies

$$c_t^i(s^t) = \theta^i \sum_{i \in I} y_t^i(s^t) \equiv \theta^i y_t(s^t),\ \forall\, i, t, s^t, \tag{1}$$

i.e. individuals consume a constant fraction of aggregate income in any period t and history s^t; and that $\theta^i = \lambda_i$ where $\lambda_i > 0$ is the Pareto weight of consumer i.

2. Let $I = 2$ and let the individual income process be

$$y_t^1 = \begin{cases} 1 \text{ if } t \text{ is even} \\ 0 \text{ if } t \text{ is odd} \end{cases}$$

$$y_t^2 = \begin{cases} 0 \text{ if } t \text{ is even} \\ 1 \text{ if } t \text{ is odd} \end{cases}$$

that is, aggregate income is always equal to 1 and individual income fluctuates deterministically between 0 and 1. Assume that consumers have log preferences.

a. Show that Pareto optimal allocations are given by:

$$c_t^1 + c_t^2 = 1,\ t = 0, 1, ...$$
$$c_t^1 / c_t^2 = \alpha / (1 - \alpha),\ t = 0, 1, ...$$

where α and $(1-\alpha)$ are the Pareto weights assigned to consumers 1 and 2, respectively.

b. Obtain the first order condition of the consumer problem:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_t \beta^t \log(c_t^i)$$

subjec to:

$$c_t^i > 0, \; t = 0, 1, ... \tag{2}$$
$$\sum_t p_t c_t^i \leq \sum_t p_t y_t^i. \tag{3}$$

and say what must be the value of the multiplier on the budget constraint (3) and the value of p_t so that this condition is identical to the first order condition of the Planner's problem, found in a.

c. Define the transfer functions $t^i(\alpha)$, for $i = 1, 2$, by

$$t^i(\alpha) = \sum_t \mu_t [c_t^i(\alpha) - y_t^i], \tag{4}$$

where μ_t is the period t multiplier on the resource constraint of the Planner's problem. $t^i(\alpha)$ is the value that the Planner would need to transfer to agent i, in excess of the value of his endowment, in order for him to consume the Pareto optimal allocation $c_t^i(\alpha)$ (found in a.). The competitive equilibrium allocation is one particular Pareto allocation, i.e. one value of α, such that transfers $t^i(\alpha)$ are zero for $i = 1, 2$. Using the condition on μ_t that ensures that the marginal conditions in the Planner's and the individual's optimization problems coincide (found in b.), find the value of α that delivers the competitive equilibrium allocation, i.e. zero net transfers $t^i(\alpha)$.

3. Consider the endowment economy studied in class, where the stochastic event $s_t \in [0, 1]$, and the probability of history s^t in period t is given by $\pi_t(s^t)$. Suppose there are two consumers and $y_t^1(s^t) = s_t$, $y_t^2(s^t) = 1 - s_t$, such that there is no aggregate uncertainty:

$$\sum_i y_t^i(s^t) = 1, \; \forall t, s^t.$$

a. Show that in a competitive equilibrium with a complete set of Arrow-Debreu securities, the equilibrium allocation of consumption is constant every period and for every possible history s^t:

$$c_t^i(s^t) = \bar{c}^i, \ \forall t, s^t, \text{ for } i = 1, 2.$$

b. Using the competitive equilibrium conditions, show that

$$\bar{c}^i = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) y_t^i(s^t),$$

i.e. each household completely smooths consumption across time and histories, and consumes the annuity value of his expected discounted endowment.

4. Ex. 8.5, 8.6, 8.7, 8.20, 8.24 in Ljunqgvist and Sargent (2018).

References:

Ljunqgvist, L. and Sargent T. J. (2018), ”Recursive Macroeconomic Theory”, fourth edition, The MIT Press.