Macroeconomics II

– Preliminary –

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João Brogueira de Sousa

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Risk with Complete Markets

- In this section, we study economies with stochastic endowments.
- We will ignore production for now, and focus on efficiency and on equilibrium outcomes of different market structures.
- We revisit time-0 and sequential trading arrangements, in economies with risk.
- Later we will study markets with incomplete markets, production, etc.

Environment

- In each period $t = 0, 1, ...$ there is a realization of a stochastic event $s_t \in S$.
- The history of events up to and including time t is denoted:

$$s^t = [s_0, s_1, ...s_t]$$

- The (unconditional) probability of a particular sequence of events s^t is:

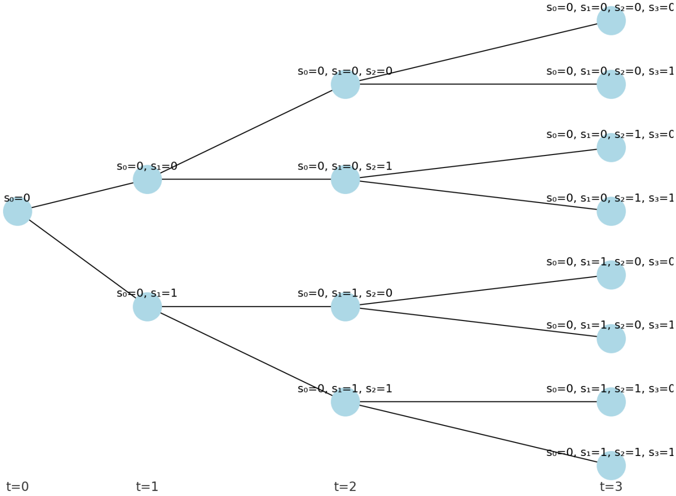
$$\pi_t(s^t)$$

- The probability of s^t conditional the realization of s^τ is:

$$\pi_t(s^t | s^\tau)$$

- Histories s^t are publicly observable.

Event Tree for $s_t \in \{0,1\}$ from $t=0$ to $t=3$ (Conditioned on $s_0=0$)



Consumers

- There are I consumers, denoted by $i = 1, ..., I$.
- Each consumer i owns a stochastic endowment of goods each period, that depends on the history s^t up to that period:

$$y_t^i(s^t)$$

- The consumer purchases a history-dependent consumption plan:

$$c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$$

- and orders these plans according to:

$$U_i(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) u_i(c_t^i(s^t)),\ 0 < \beta < 1.$$

- u_i is an increasing, twice cont. diff., str. concave function, with

$$\lim_{c \to 0} u_i'(c) = +\infty$$

Feasibility

- In this economy, feasible allocations satisfy

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t) \tag{1}$$

for all t and all s^t.

Efficient allocations

- Pareto Optimal allocations are the solution to the following problems:

$$\max \sum_{i=1}^{I} \lambda_i U_i(c^i) \tag{2}$$

subject to

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \ \forall t, s^t \tag{3}$$

for some nonnegative Pareto weights $\lambda_i, i = 1, ..., I$.

- Different $\{\lambda_i\}_{i=1}^{I}$: different point on the *Pareto Frontier*.

Pareto Problem

- Form the Lagrangian:

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_{i=1}^{I} \lambda_i \beta^t \pi_t(s^t) u_i(c_t^i(s^t)) + \theta_t(s^t) \sum_{i=1}^{I} \left[y_t^i(s^t) - c_t^i(s^t) \right] \right\}$$

- Note the multipliers: $\theta_t(s^t)$.
- Resource constraint must hold for each period t and history s^t.
- First order condition with respect to $c_t^i(s^t)$:

$$\lambda_i \beta^t \pi_t(s^t) u_i'(c_t^i(s^t)) = \theta_t(s^t) \tag{4}$$

- Implies, for consumer 1 and $\forall i$:

$$\frac{u_i'(c_t^i(s^t))}{u_1'(c_t^1(s^t))} = \frac{\lambda_1}{\lambda_i}, \ \forall t, s^t. \tag{5}$$

Pareto Allocations

$$\frac{u_i'(c_t^i(s^t))}{u_1'(c_t^1(s^t))} = \frac{\lambda_1}{\lambda_i},\ \forall t, s^t. \tag{6}$$

- The above is equivalent to:

$$c_t^i(s^t) = u_i'^{-1}(\lambda_i^{-1}\lambda_1 u_1'(c_t^1(s^t))),\ \forall t, s^t. \tag{7}$$

- Using the resource constraint:

$$\sum_i u_i'^{-1}(\lambda_i^{-1}\lambda_1 u_1'(c_t^1(s^t))) = \sum_i y_t^i(s^t) \tag{8}$$

- One condition in one unknown, $c_t^1(s^t)$.
- Given $\{\lambda_i\}_{i=1}^I$, $c_t^i(s^t)$ depends only on aggregate endowment, $\sum_i y_t^i(s^t)$, not on the individual i or the distribution of $y_t^i(s^t)$.

Pareto Allocations

In this economy, a Pareto Allocation is a function of the realized aggregate endowment and does not depend separately on the specific history s^t or on the cross-section distribution of individual endowments realized at any period t:

$$c_t^i(s^t) = c_\tau^i(\tilde{s}^\tau), \text{ for } s^t, \tilde{s}^\tau \text{ such that } \sum_i y_t^i(s^t) = \sum_i y_\tau^i(\tilde{s}^\tau)$$

Note also that only ratio λ_i/λ_j affects the allocation, so we can normalize, e.g. $\sum_i \lambda_i = 1$.

Examples

- 2 households (or 2 types of households)
- 2 possible states: $s_t \in \{H, T\}$.
- One period, one consumption good, c.
- Each household i is endowed with a state contingent endowment: y^i_H, y^i_T.
- Each household i has a utility function which satisfies the Expected Utility Hypothesis:

$$U_i(c^i) = \pi_H u_i(c^i_H) + \pi_T u_i(c^i_T)$$

- π_H (π_T) is the probability of state H (T).

Examples

Pareto Problem:

$$\max_{\{c_T^i, c_H^i\}_{i=1,2}} \sum_{i=1,2} \lambda_i \left[\pi_H u_i(c_H^i) + \pi_T u_i(c_T^i) \right]$$

with $\lambda_i > 0$, subject to:

$$\sum_i c_H^i = \sum_i y_H^i$$
$$\sum_i c_T^i = \sum_i y_T^i$$

Example 1: Suppose there is no aggregate uncertainty:

$$\sum_i y_H^i = \sum_i y_T^i.$$

Claim: If u^i is strictly concave, Pareto Optimal allocations have Full Insurance:

$$c_H^i = c_T^i, \quad \forall i.$$

Proof: use Jensen's Inequality.

Examples

Pareto Problem:

$$\max_{\{c^i_T, c^i_H\}_{i=1,2}} \sum_{i=1,2} \lambda_i \left[\pi_H u_i(c^i_H) + \pi_T u_i(c^i_T) \right]$$

with $\lambda_i > 0$, subject to:

$$\sum_i c^i_H = \sum_i y^i_H$$
$$\sum_i c^i_T = \sum_i y^i_T$$

Example 2: Suppose there is aggregate uncertainty:

$$\sum_i y^i_H \neq \sum_i y^i_T.$$

FOC (with multipliers μ_H, μ_T):

$$\lambda_i \pi_H u_i'(c^i_H) = \mu_H,\ i = 1, 2;$$
$$\lambda_i \pi_T u_i'(c^i_T) = \mu_T,\ i = 1, 2.$$

Example 2

FOC (with multipliers μ_H, μ_T):

$$\lambda_i \pi_H u_i'(c_H^i) = \mu_H, \; i = 1, 2; \tag{9}$$

$$\lambda_i \pi_T u_i'(c_T^i) = \mu_T, \; i = 1, 2. \tag{10}$$

Suppose $\lambda_1 = \lambda_2$ and $u_1 = u_2 = u$. Then:

$$c_H^1 = c_H^2 \equiv c_H, \; c_T^1 = c_T^2 \equiv c_T. \tag{11}$$

$$c_H = \sum_i y_H^i/2, c_T = \sum_i y_T^i/2 \tag{12}$$

Suppose u is homogeneous: $u(\theta c) = \theta^\eta u(c)$ for some η. Then

$$u'(c) = \eta u(c)/c$$
$$u(c) = u(1)c^\eta$$
$$\implies u'(c) = u(1)\eta c^{\eta-1}$$

Example 2

Using the FOCs with $\lambda_1 = \lambda, \lambda_2 = 1 - \lambda$:

$$\lambda (c_H^1)^{\eta-1} = (1-\lambda)(c_H^2)^{\eta-1} \tag{13}$$
$$\lambda (c_T^1)^{\eta-1} = (1-\lambda)(c_T^2)^{\eta-1} \tag{14}$$

or

$$c_s^1 / c_s^2 = \left(\frac{1-\lambda}{\lambda}\right)^{\frac{1}{\eta - 1}},\ s = H, T. \tag{15}$$

HH 1 gets a (constant) multiple of what 2 gets, in both states.

$$c_s^1 = \phi(y_s^1 + y_s^2), \quad c_s^2 = (1-\phi)(y_s^1 + y_s^2),\ s = H, T. \tag{16}$$

$\phi \in (0, 1)$ traces the Pareto Frontier. No Full Insurance.

Competitive Equilibrium

- The endowment economy is the same as above (Example 2).
- There is a market for state contingent claims on the consumption good.
- $q(s)$ is the price of a claim on 1 unit of c in state s, $s \in \{H, T\}$, and zero otherwise.
- Consumers trade claims on c before the state s is realized.
- Individual problem:

$$\max_{c_s^i} \sum_{s=H,T} \pi_s u_i(c_s^i) \tag{17}$$

subject to:

$$\sum_{s=H,T} q(s) c_s^i \leq \sum_{s=H,T} q(s) y_s^i, \tag{18}$$

taking $q(s)$ as given.
- Feasible allocation:

$$\sum_{i=1,2} c_s^i \leq \sum_{i=1,2} y_s^i, \; s = H, T. \tag{19}$$

Competitive Equilibrium

Definition: A competitive equilibrium is a feasible allocation (19), prices $q(H), q(T)$, such that, given prices, the allocation solves each consumer's problem (17)-(18).

Competitive Equilibrium Allocations

- FOC of the consumer i's problem:

$$\pi_s u_i'(c_s^i) - \mu^i q(s) = 0, s = H, T. \tag{20}$$

- μ^i: Lagrange multiplier on i's budget constraint.
- These imply:

$$\frac{u_1'(c_s^1)}{u_2'(c_s^2)} = \frac{\mu^1}{\mu^2},\ s = H, T \tag{21}$$

or

$$c_s^2 = {u_2'}^{-1}\left[\frac{\mu^2}{\mu^1} u_1'(c_s^1)\right] \tag{22}$$

- The resource constraint implies:

$$c_s^1 + {u_2'}^{-1}\left[\frac{\mu^2}{\mu^1} u_1'(c_s^1)\right] = \sum_i y_s^i \tag{23}$$

- c_s^i depends only on aggregate endowment (RHS) and on $\frac{\mu^2}{\mu^1}$.

Competitive Equilibrium Allocations

- Recall the FOC of the Pareto Problem, (9)-(10):

$$\frac{u_1'(c_s^1)}{u_2'(c_s^2)} = \frac{\lambda^2}{\lambda^1},\ s = H, T \tag{24}$$

  where λ_i is the Pareto weight on consumer i.

- Competitive Equilibrium allocation is one particular Pareto Optimal allocation, with:

$$\lambda_i = \mu_i^{-1},\ i = 1, 2.$$

- First Welfare Theorem.
- Applies more generally, for infinite horizon economy with $s^t = [s_0, s_1, ..., s_t]$.

Equilibrium with Complete Markets

- Consumers trade a complete set of history-contingent claims to consumption.
- Trade at $t = 0$, claims on time t, history s^t, consumption, at price $q_t(s^t)$
- Consumer's budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) y_s^i(s^t). \tag{25}$$

- C's problem is to choose c^i to maximize

$$U^i(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) u_i(c_t^i(s^t)) \tag{26}$$

subject to (25), taking $q_t(s^t)$ as given.

Equilibrium with Complete Markets

- Lagrange multiplier on consumer i's budget: μ_i
- FOC:

$$\frac{\partial U^i(c^i)}{\partial c_t^i(s^t)} = \beta^t \pi_t(s^t) u_i'(c_t^i(s^t)) = \mu_i q_t(s^t) \tag{27}$$

- For Consumer i and j:

$$\frac{u_i'(c_t^i(s^t))}{u_i'(c_t^j(s^t))} = \frac{\mu_i}{\mu_j},\ \forall t, s^t. \tag{28}$$

- This implies, e.g. for $i = 1$:

$$c_t^j(s^t) = u_j'^{-1}(\mu_1^{-1}\mu_j u_1'(c_t^1(s^t))),\ \forall t, s^t. \tag{29}$$

- Using the resource constraint:

$$\sum_j u_j'^{-1}(\mu_1^{-1}\mu_j u_1'(c_t^1(s^t))) = \sum_j y_t^j(s^t) \tag{30}$$

$\implies c_t^i(s^t)$ depends on aggregate endowment and on $\frac{\mu_j}{\mu_i}$'s.

Optimality of equilibrium allocation

- A CE allocation is a particular Pareto optimal allocation, with Pareto weights:

$$\lambda_i = \mu_i^{-1}.$$

- Furthermore, if $\theta_t(s^t)$ are the multipliers on resource constraint at t, s^t, then (recall (4))

$$\lambda_i \beta^t \pi_t(s^t) u_i'(c_t^i(s^t)) = \theta_t(s^t) \quad (31)$$

$$\implies \theta_t(s^t) = q_t(s^t) \quad (32)$$

- Shadow price $\theta_t(s^t)$ of the planning problem equal to CE prices $q_t(s^t)$.

CE Solution

- Algorithm to solve for allocation and prices? Negishi (1960):
  1. Set $\mu_1 > 0$.
  2. Guess $\mu_j > 0$ for $j = 2, ..., I$. Solve

$$\sum_j u_j'^{-1}(\mu_1^{-1}\mu_j u_1'(c_t^1(s^t))) = \sum_j y_t^j(s^t) \qquad (33)$$

     for $c_t^1(s^t)$ and then for $c_t^i(s^t)$, for $i = 2, ..., I$, using (28).
  3. Use FOC of any i to obtain prices:

$$\beta^t \pi_t(s^t) u_i'(c_t^i(s^t)) = \mu_i q_t(s^t) \qquad (34)$$

  4. For $i = 2, ..., I$, check the budget constraint under the prices and allocation found in 2 and 3. If the cost of consumption exceeds the value of the endowment for consumer i, raise μ_i. Otherwise decrease μ_i.
  5. Iterate on 2-4 until convergence.

CRRA Preferences

- Consider the CRRA case:

$$u_i(c) = c^{1-\gamma}/(1-\gamma),\ \gamma > 0. \tag{35}$$

- Then (28) becomes:

$$c_t^i(s^t) = c_t^j(s^t)\left(\frac{\mu_i}{\mu_j}\right)^{-1/\gamma} \tag{36}$$

- Consumption of different agents are a constant fraction of one another for all t, s^t.
- Individual consumption is perfectly correlated with aggregate endowment:

$$c_t^i(s^t) = \alpha_i \sum_i y_t^i(s^t) = \alpha_i c_t(s^t) \tag{37}$$

- Aggregate consumption: $c_t(s^t)$. i's consumption share: α_i.

CRRA: Asset pricing

- With CRRA preferences:

$$\log\left(\frac{c^i_{t+1}(s^{t+1})}{c^i_t(s^t)}\right) = \log\left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)}\right) \tag{38}$$

- And from the Euler equation:

$$\begin{aligned}\frac{q_{t+1}(s^{t+1})}{q_t(s^t)} &= \beta\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}\left(\frac{c^i_{t+1}(s^{t+1})}{c^i_t(s^t)}\right)^{-\gamma} \\ &= \beta\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}\left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)}\right)^{-\gamma}\end{aligned}$$

- Equilibrium prices can be written as functions of aggregate consumption only.
- Consumption-based asset pricing literature, developed in Lucas (1978).

Other Assets

- The Arrow-Debreu securities are enough to complete the market (price all t, s^t goods).
- However we can price any redundant asset using the equilibrium prices $q_t(s^t)$.
- Consider an asset that pays the stream of dividends:

$$\{d_t(s^t)\}_{t=0}^{\infty}$$

  time t history s^t consumption goods.
- The price of this asset as of time 0 is:

$$p_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) d_t(s^t) \tag{39}$$

Other Assets

- The price of a riskless consol, i.e. $d_t(s^t) = 1$, for all t and s^t:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) \tag{40}$$

- The price of a riskless strip, i.e. $d_\tau(s^\tau) = 1$ for $t = \tau$ and all s^τ, and 0 otherwise:

$$\sum_{s^t} q_t(s^t) \tag{41}$$

Tail Assets

- Take the dividend stream $\{d_t(s^t)\}_{t=0}^{\infty}$
- The price of the $\tau \geq t$ remaining dividend flows, conditional on history s^t at time t:

$$p(s^t) = \sum_{\tau \geq t} \sum_{s^\tau | s^t} q_\tau(s^\tau) d_\tau(s^\tau) \tag{42}$$

- In units of period t history s^t goods:

$$p_t(s^t) = \sum_{\tau \geq t} \sum_{s^\tau | s^t} \frac{q_\tau(s^\tau)}{q_t(s^t)} d_\tau(s^\tau) \tag{43}$$

Arrow Securities

- Alternative market structure: trade in one-period-ahead state contingent consumption claims
- At each date $t \geq 0$, consumers trade claims to date $t+1$ consumption, whose payment is contingent on the realization of s^{t+1}.
- Also known as 'Arrow securities' (Arrow (1964)).
- Claims to time t history s^t, consumption goods, other than $y_t^i(s^t)$, of consumer i:

$$\tilde{a}_t^i(s^t)$$

- Price of one unit of $t+1$ consumption good, contingent on the realization of s_{t+1} at $t+1$, after history s^t:

$$\tilde{Q}_t(s_{t+1}|s^t)$$

Sequential trading

- The problem of consumer i is:

$$\max_{\tilde{c}_t^i(s^t),\{\tilde{a}_{t+1}^i(s^{t+1})\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) u_i(\tilde{c}_t^i(s^t)) \tag{44}$$

subject to

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s^{t+1}) \tilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s^t) \tag{45}$$

$$-a_{t+1}^i(s^{t+1}) \leq \tilde{A}_{t+1}(s^{t+1}) \tag{46}$$

- $\{\tilde{a}_{t+1}^i(s^{t+1})\}$ is a vector of claims to $t+1$ state s_{t+1} goods.
- Vector size is $\#s_{t+1}$ (states in $t+1$).
- Why do we need $\tilde{A}_{t+1}(s^{t+1})$ here? no-Ponzi condition.
- A "natural" debt limit:

$$\tilde{A}_{t+1}(s^{t+1}) = \sum_{\tau \geq t} \sum_{s^\tau | s^t} q_\tau(s^\tau) y_\tau^i(s^\tau) \tag{47}$$

Competitive Equilibrium with Sequential Trades

Definition: A distribution of wealth is a vector $\vec{\tilde{a}}_t(s^t) = \{\tilde{a}^i_t(s^t)\}_{i=1}^I$ such that $\sum_i a^i_t(s^t) = 0$.

Definition: A competitive equilibrium with sequential trading of one-period Arrow securities is an initial distribution of $\vec{\tilde{a}}_0(s_0)$, a vector of borrowing limits $\{A^i_t(s^t)\}$ for all i, t and s^t, a feasible allocation $\{\tilde{c}^i\}_{i=1}^I$, prices $\tilde{Q}_t(s_{t+1}|s^t)$ such that: (i) given prices, the initial wealth distribution and the natural debt limits for all i, the consumption allocation $\tilde{c}^i$ and the portfolio $\{\tilde{a}^i_{t+1}(s^{t+1})\}$ solves the consumer problem for all i, and (ii) for all $\{s^t\}$, allocations and portfolios satisfy

$$\sum_i \tilde{c}^i_t(s^t) = \sum_i y^i_t(s^t) \tag{48}$$

and

$$\sum_i \tilde{a}^i_{t+1}(s^{t+1}) = 0 \tag{49}$$

Arrow Securities: Prices

- From the consumer's FOC:

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u_i'(\tilde{c}^i_{t+1}(s^{t+1}))}{\tilde{u}_i'(c^i_t(s^t))} \pi_t(s^{t+1}|s^t), \quad \text{for all } t, s^t, s_{t+1}. \tag{50}$$

- Note that this condition is equivalent to (27), and holds with the same allocation if:

$$Q_t(s_{t+1}|s^t) = \frac{q_{t+1}(s^{t+1})}{q_t(s^t)} \tag{51}$$

- Equilibrium allocations coincide if $\vec{\tilde{a}}_0(s_0) = 0$.

Recursive competitive equilibrium

- Suppose that the state of the economy is Markovian:

$$\pi_{t+1}(s^{t+1}|s^t) = \pi_{t+1}(s^{t+1}|s_t) = \pi(s_{t+1}|s_t)\pi(s_t|s_{t-1})...\pi(s_1|s_0)$$

- The endowment process only depends on the state in period t:

$$y_t^i(s^t) = y^i(s_t), \text{ for all } i.$$

- All previous results hold, but since aggregate endowment is a function of s_t:

$$c_t^i(s^t) = c^i(s_t) \tag{52}$$

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u_i'(\tilde{c}^i(s_{t+1}))}{\tilde{u}_i'(c^i(s_t))} \pi_t(s_{t+1}|s^t) \equiv Q(s_{t+1}|s_t) \tag{53}$$

Recursive formulation

- Denote current realization s_t by s, next period's by s'.
- Endowments $y^i(s)$.
- Prices $Q(s'|s)$
- Consumer i state at time t is: wealth a_t, and the current realization s_t.
- Policy functions (decisions):

$$c_t^i = c(a, s)$$
$$a_{t+1}^i = a'(a, s)$$

- Let $v^i(a, s)$ denote the optimal value of consumer i's problem starting from state (a, s).

Recursive formulation

- Bellman equation for $v^i(a, s)$:

$$v^i(a, s) = \max_{c, a(s')} \left\{ u_i(c) + \beta \sum_{s'} \pi(s'|s) v^i(a(s'), s') \right\} \quad (54)$$

subject to

$$c + \sum_{s'} Q(s'|s) a(s') \leq y^i(s) + a, \quad (55)$$

$$c \geq 0, \quad (56)$$

$$-a(s') \leq A^i(s'), \ \forall s'. \quad (57)$$

Recursive Competitive Equilibrium

Definition: A recursive competitive equilibrium is an initial distribution of $\vec{\vec{a}}_0$, a vector of borrowing limits $\{A^i(s)\}_{i=1}^{I}$ for all s, prices $Q(s'|s)$, value functions $\{v^i(a,s)\}_{i=1}^{I}$, and policy functions $\{c^i(a,s), a^{i\prime}(a,s)\}_{i=1}^{I}$, such that:

(i) The borrowing limits satisfy:

$$A^i(s) = y^i(s) + \sum_{s'} Q(s'|s) A^i(s'|s), \tag{58}$$

(ii) For all i, given a_0^i, $A^i(s)$ and prices $Q(s'|s)$, the value function and the policy rules solve the consumer's problem;

(iii) For all realizations of $\{s_t\}_{t=0}^{\infty}$, the consumption allocation $c^i(s_t)$ and the portfolio $\{a_{t+1}^i(s')\}$ implied by the policy rules satisfy $\sum_i c^i(s_t) = \sum_i y_t^i(s_t)$ and $\sum_i a_{t+1}^i(s') = 0$.