Exercises

1. Vector Autoregression (VAR)

1. Suppose you have a stylized model of the very short run (when prices are sticky) that suggests that the level of output is determined by

$$y_t = \gamma\epsilon_t^d + \theta\epsilon_{t-1}^d - \beta i_t \tag{1}$$

where $\epsilon_t^d \sim i.i.d.(0, \sigma_d^2)$ is a demand shock and i_t is the interest rate controlled by the central bank as follows:

$$i_t = \phi_y y_t + \epsilon_t^m \tag{2}$$

and $\epsilon_t^m \sim i.i.d.(0, \sigma_m^2)$ is a shock on monetary policy productivity. The shocks are orthogonal.
Since you have data on y_t and i_t, you decide to estimate a SVAR and identify the shocks

a) IRFS:

i) Discuss the interpretation of Impulse Response Functions (IRFs) in a VAR context.
Solution:
Impulse Response Functions (IRFs) in a Vector Autoregression (VAR) model provide insights into the dynamic interactions among the variables included in the system. IRFs show the response of each variable in the VAR model to a one-unit shock to one of the variables, holding all other variables constant. IRFs also show the response of other variables in the system to a shock in a particular variable, providing insights into the dynamic interconnections among the variables.

ii) Find the effect of a unitary change in ϵ_t^m on y_t and i_t for $t \geq 0$
Solution:
First, we can write Equations as:

$$y_t = \tilde{\beta}\gamma\epsilon_t^d + \tilde{\beta}\theta\epsilon_{t-1}^d - \tilde{\beta}\beta\epsilon_t^m$$
$$i_t = \tilde{\beta}\left[\phi_y(\gamma\epsilon_t^d + \theta\epsilon_{t-1}^d) + \epsilon_t^m\right]$$

Unitary change of ϵ_t^m:

- For output: $\frac{\partial y_{t+h}}{\partial e_t^m}$ for $h \geq 0$: $\begin{cases} h = 0: & -\tilde{\beta}\beta \\ h \geq 0: & 0. \end{cases}$
- For interest rate: $\frac{\partial i_{t+h}}{\partial e_t^m}$ for $h \geq 0$: $\begin{cases} h = 0: & \tilde{\beta} \\ h \geq 0: & 0. \end{cases}$

iii) Find the effect of a unitary change in ϵ_t^d on y_t and i_t for $t \geq 0$

Solution:

Unitary change of ϵ_t^d:

- For output: $\frac{\partial y_{t+h}}{\partial e_t^m}$ for $h \geq 0$: $\begin{cases} h=0: & \tilde{\beta}\gamma \\ h=1: & \tilde{\beta}\theta \\ h \geq 1: & 0. \end{cases}$
- For interest rate: $\frac{\partial i_{t+h}}{\partial e_t^m}$ for $h \geq 0$: $\begin{cases} h=0: & \tilde{\beta}\phi_y\gamma \\ h=1: & \tilde{\beta}\phi_y\theta \\ h \geq 1: & 0. \end{cases}$

iv) Is y_t stationary? Is it ergodic?

Solution:

A serie is Covariance Stationary if:

A. Constant Mean:

$$E(y_t) = -\beta E(i_t) = -\beta\gamma E(y_t) \Leftrightarrow$$
$$E(y_t)(1+\beta\gamma) = 0 \Leftrightarrow E(y_t) = 0$$

B. Constant Variance:

$$Var(y_t) = \tilde{\beta}^2\gamma^2 Var(\epsilon_t^d) + \tilde{\beta}^2\theta^2 Var(\epsilon_{t-1}^d) + \tilde{\beta}^2\beta^2 Var(\epsilon_t^m) \Leftrightarrow$$
$$Var(y_t) = \tilde{\beta}^2\Big[(\gamma^2+\theta^2)\sigma_d^2 + \beta^2\sigma_m^2\Big] = \gamma_0$$

C. Covariance converge to a constant:

$$\begin{aligned} \gamma_1 =& Cov(y_t, y_{t-1}) = \\ & E\Big[\tilde{\beta}\big(\gamma\epsilon_t^d + \theta\epsilon_{t-1}^d - \beta\epsilon_t^m\big), \tilde{\beta}\big(\gamma\epsilon_{t-1}^d + \theta\epsilon_{t-2}^d - \beta\epsilon_{t-1}^m\big)\Big] \\ & = \tilde{\beta}^2\theta\gamma\sigma_d^2 \\ \gamma_h =& Cov(y_t, y_{t-h}) \text{ for } h \geq 1 = 0 \end{aligned}$$

Is it Ergodic? Let $\{y_t\}$ be a covarinace stationary process with constant meand and constant autocovariances.

If $\sum_{j=0}^{\infty} |\gamma_j| < \infty$ then $\{y_t\}$ is ergodic for the mean.

$$\begin{aligned} \sum_{j=0}^{\infty} |\gamma_j| =& \tilde{\beta}^2\Big[(\gamma^2+\theta^2)\sigma_d^2 + \beta^2\sigma_m^2\Big] + \tilde{\beta}^2\theta\gamma\sigma_d^2 = \\ & = \tilde{\beta}^2\Big[(\gamma^2+\theta^2+\theta\gamma)\sigma_d^2 + \beta^2\sigma_m^2\Big] \end{aligned}$$

b) Represent the previous model as a Vector Moving Average (VMA) (Wold representation). (hint: $X_t = D(L)U_t$, where $X_t = \begin{bmatrix} y_t \\ i_t \end{bmatrix}$ and $U_t = \begin{bmatrix} \epsilon_t^d \\ \epsilon_t^m \end{bmatrix}$

Solution:

Let's right the system as $AX_t = BU_t$.

$$\begin{cases} y_t + \beta i_t = \gamma \epsilon_t^d + \theta \epsilon_{t-1}^d \\ i_t - \phi_y y_t = \epsilon_t^m \end{cases}$$

$$\begin{bmatrix} 1 & \beta \\ -\phi_y & 1 \end{bmatrix} \begin{bmatrix} y_t \\ i_t \end{bmatrix} = \begin{bmatrix} \gamma + \theta L & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t^d \\ \epsilon_t^m \end{bmatrix}$$

Assume that $D(L) = A^{-1}B$. Then, we need the inverse of A.

$$A^{-1} = \frac{1}{1 + \phi_y \beta} \begin{bmatrix} 1 & -\beta \\ \phi_y & 1 \end{bmatrix} = \tilde{\beta} \begin{bmatrix} 1 & -\beta \\ \phi_y & 1 \end{bmatrix}$$

So, we can write the VMA as:

$$X = \begin{bmatrix} \tilde{\beta}(\gamma + \theta L) & \tilde{\beta}\beta \\ \phi_y(\gamma + \theta L)\tilde{\beta} & \tilde{\beta} \end{bmatrix} U$$

c) What are the conditions for the VMA to be written in VAR(∞) form and the shocks to be fundamental? (Be as specific as you can).

Solution:

A VMA is invertible if and only if the roots of the polynomial $|D(z)| = 0$ lie outside the unit circle: $|z| > 1$.

$$\left| \begin{bmatrix} \gamma + \theta z & 0 \\ \phi_y\,(\gamma + \theta z) & 1 \end{bmatrix} \right| = 0 \Leftrightarrow (\gamma + \theta z) = 0 \Leftrightarrow z = -\frac{\gamma}{\theta}$$

In order to the shocks to be fundamental:
$|z| > 1 \Leftrightarrow \left| -\frac{\gamma}{\theta} \right| > 1 \Leftrightarrow \gamma > \theta$

d) A strong response of the Central Bank to output affect the possibility for the VMA to be fundamental?

Solution:

No. The fundamentalness of the VMA only depends on how output changes with respect to demand shocks.

e) Should the output response to ϵ^d be stronger at time t or at time $t+1$ for the VMA to be fundamental?

Solution:

As concluded before, for the VMA to be fundamental, it must be that the response of output to time $t+1$ to be bigger than to time t

$$y_{t+1} = \gamma \epsilon_{t+1}^d + \theta \epsilon_t^d$$

2. Explain the concept of Forecast Variance Decomposition in VAR models.

Solution:

FVD quantifies the proportion of the forecast error variance of each variable that can be attributed to its own shocks versus the shocks of other variables in the system.

The portion of forecast error variance attributed to a variable's own shocks

indicates the extent to which the variable's future values are driven by its own dynamics.
The proportion of forecast error variance attributed to other variables' shocks reflects the degree of interdependence among the variables in the VAR model. Higher values suggest stronger interconnections.

3. Define a Vector Error Correction Models (VECM)
   Solution:
   A Vector Error Correction Model (VECM) is a type of econometric model used to analyze the long-run relationship among multiple time series variables. It is an extension of the Vector Autoregression (VAR) model that accounts for cointegration among the variables.