Exercises Week 5

1. VECM Models

1. Explain the concept of cointegration. What distinguishes a cointegrated series from a spurious regression?
   Solution:
   Cointegration refers to a statistical property of time series variables which, while individually nonstationary (typically $I(1)$), exhibit a stable long-run relationship. Specifically, if a linear combination of nonstationary variables is stationary ($I(0)$), the variables are said to be cointegrated. This contrasts with spurious regression, where relationships between integrated variables can appear significant due to shared stochastic trends, despite no actual economic link.

2. Consider two integrated time series $y_{1,t}$ and $y_{2,t}$. Define formally what it means for these series to be cointegrated. Include conditions for cointegration.
   Solution:
   Let $y_{1,t}$ and $y_{2,t}$ be two time series that are both $I(1)$. These series are cointegrated if there exists a vector $\beta = (\beta_1, \beta_2)$ such that:

$$\beta_1 y_{1,t} + \beta_2 y_{2,t} \sim I(0), \quad \text{with } \beta \neq 0.$$

   More generally, a vector $y_t \in \mathbb{R}^n$ is cointegrated of order $CI(d, b)$ if all elements are $I(d)$, and there exists a vector β such that $\beta' y_t \sim I(d-b)$. Most economic applications focus on $CI(1, 1)$.

3. Write down the mathematical form of a simple Vector Error Correction Model (VECM) for two cointegrated variables and explain the role of adjustment parameters.
   Solution:
   For two cointegrated variables, a simple Vector Error Correction Model (VECM) is written as:

$$\begin{aligned}
\Delta y_{1,t} &= \alpha_1 (y_{1,t-1} + \beta_2 y_{2,t-1}) + \varepsilon_{1,t} \\
\Delta y_{2,t} &= \alpha_2 (y_{1,t-1} + \beta_2 y_{2,t-1}) + \varepsilon_{2,t}
\end{aligned}$$

   Here, α_1 and α_2 are the adjustment coefficients indicating how each variable responds to disequilibrium. The term $y_{1,t-1} + \beta_2 y_{2,t-1}$ is the error correction term. A stable long-run relationship generally requires $\alpha_1 \leq 0$ and $\alpha_2 \geq 0$, and at least one of them nonzero.

4. Describe Johansen's Methodology for testing cointegration. Outline the main steps involved and explain the use of trace and maximum eigenvalue tests.
Solution:
Johansen's approach to cointegration testing relies on maximum likelihood estimation of a VAR model rewritten in its VEC form:

$$\Delta y_t = \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + \cdots + \Gamma_{p-1} \Delta y_{t-p+1} + \varepsilon_t,$$

where $\Pi = \alpha\beta'$ contains information on cointegration. The number of cointegrating vectors (i.e., the rank of Π) is determined by two tests:

- **Trace test:**

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^{n} \log(1 - \hat{\lambda}_i)$$

tests $H_0 : \text{rank}(\Pi) \leq r$ vs. $H_1 : \text{rank}(\Pi) > r$

- **Maximum eigenvalue test:**

$$\lambda_{\max}(r) = -T \log(1 - \hat{\lambda}_{r+1})$$

tests $H_0 : \text{rank}(\Pi) = r$ vs. $H_1 : \text{rank}(\Pi) = r + 1$

These tests are applied sequentially to determine the number of cointegrating relations. Johansen's method is preferred over Engle-Granger due to its ability to test multiple cointegrating vectors and impose/examine restrictions.

2. VECM Estimation - Computational Problem

The Central Bank of Macronova is analyzing the country's macroeconomic environment. They have collected historical data on GDP and Consumption to assess whether a stable long-term relationship exists between these two crucial variables.

Using the provided dataset:

1. Perform Johansen's test for cointegration between GDP and Consumption. Report and interpret the results.
2. Estimate a Vector Error Correction Model (VECM) for the data and interpret the estimated adjustment coefficients.
3. Provide a 5-period forecast for both variables using your estimated VECM. Interpret your results in the context of economic policy.

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