1.

(v) If the sequence $\{p^n\}_{n=0}^{\infty}$ in $\mathbb{R}^L_{++}$ converges to $p \in \mathbb{R}^L_+ \setminus \{0\}$, $p_\ell = 0$ for some ℓ,
Then $\operatorname{Max}\{z_1(p^n), \dots, z_L(p^n)\} \to \infty$

<u>Claim</u>: There exists at least one agent h s.t. $p(\omega^h + \sum_j \Theta_j^h y_j(p)) > 0$
(i.e. at prices p, some consumer has positive wealth).

<u>Proof</u>: Assume not. Then, for all h, $p\omega^h + \sum_j \Theta_j^h p y_j(p) = 0$
($p \geq 0$, $\omega^h \geq 0$, profits are non-negative – otherwise, $y_j(p) = 0$ would be optimal). Summing over h yields $p\sum_h \omega^h + \sum_h \sum_j \Theta_j^h p y_j(p) = 0 \Leftrightarrow$

$$\Leftrightarrow p\sum_h \omega^h + \sum_j p y_j(p) \underbrace{\sum_h \Theta_j^h}_{1} = 0 \Leftrightarrow \underbrace{p\sum_h \omega^h}_{\geq 0} + \underbrace{\sum_j p y_j(p)}_{\geq 0} = 0 \Rightarrow$$

$$\Rightarrow p\sum_h \omega^h = 0 \text{ and } \sum_j p y_j(p) = 0.$$

i.e. all the goods for which there is a positive endowment have 0 price and all the firms are making zero profits.

But we know that $\exists \bar{x} >> 0$ s.t. $\bar{x} \in \{\sum_h \omega^h\} + Y$ by assumption.

So $p\bar{x} > 0$ and $p\bar{x} = \underbrace{p\sum_h \omega^h}_{0} + p\bar{y}$ where $\bar{y} = \sum \bar{y}_j$ and $p\bar{y} > \sum_j p y_j(p)$

which contradicts the fact that firms are maximizing profits for prices p.

Going back to the proof of (v), we can consider an agent with positive wealth at prices p. Assume that $\operatorname{Max}\{z_1^h(p^n), \dots, z_L^h(p^n)\} \not\to \infty$. Since Y_j is bounded above, $y_{j\ell}(p^n) \not\to \infty$; so for $z_\ell^h(p^n) = x_\ell^h(p^n) - \omega_\ell^h - \sum_j \Theta_j^h y_{j\ell}(p^n)$ not to converge to infinity, we must have that $x_\ell^h(p^n)$ cannot converge to infinity either.

But then an infinite subsequence of the sequence $x^h(p^n)$ is bounded and a subsequence of this subsequence converges to a bundle $\bar{x}$. Assume w.l.o.g. that $x^n \to \bar{x}$.

We know that $\bar{x} \in B^h(p)$. Let $\hat{x} \in B^h(p)$. For any such $\hat{x}$, we can write it as the limit of $(\hat{x}^n)$, $\hat{x}^n \in B^h(p^n)$. Since $u^h(\hat{x}^n) \leq u^h(x^n)$, we infer that $u^h(\hat{x}) \leq u^h(\bar{x})$.

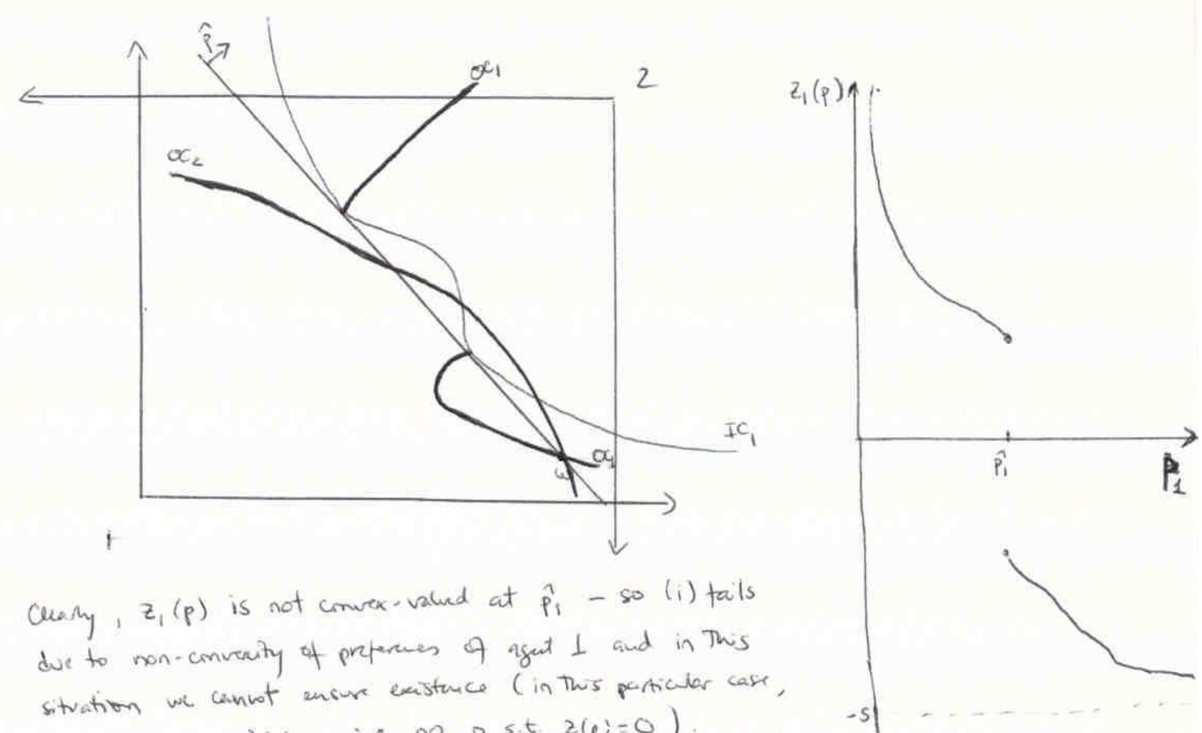
But if we choose $\hat{x}$ to have an infinite amount of good ℓ s.t. $p_\ell = 0$, and the same quantities of goods with positive prices as $\bar{x}$, then $u^h(\hat{x}) > u^h(\bar{x})$ by strong monotonicity and we reach a contradiction.

2.

17.C.6 $L=2$.

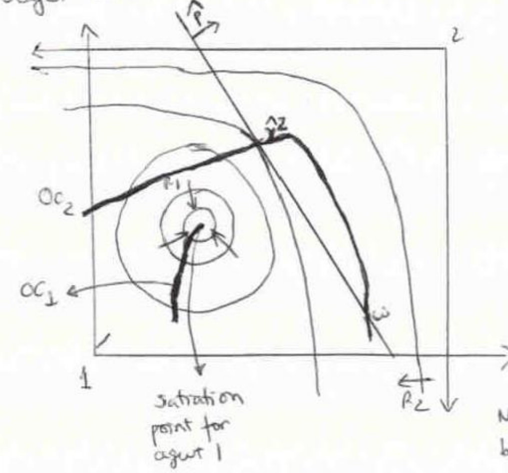
Note: It is easy to find mathematical functions $z(p)$ that fail the assumptions of Prop 17.B.2. The interesting part of this exercise is discovering which assumptions on preferences or endowments must be met so that the assumptions on $z(p)$ still hold or, alternatively, which assumptions on the underlying economic problem must be relaxed so that the aggregate excess demand function $z(p)$ will fail to meet (i) or (iii) or (iv).

→ (i) continuity of $z(p)$ can fail because the actual preferences are not continuous (Think of lexicographic preferences) or because preferences aren't convex; Looking at the graph, $z(p)$ is no longer a function but a correspondence and the equivalent statement of (i) for a correspondence would be that $z(p)$ be upper hemicontinuous and convex-valued.



Clearly, $z_1(p)$ is not convex-valued at $\hat{p}_1$ – so (i) fails due to non-convexity of preferences of agent 1 and in this situation we cannot ensure existence (in this particular case, there is no equilibrium, i.e. no p s.t. $z(p)=0$).
Notice, however, (iii) holds (agents are always on their budget constraints) and (iv) also holds since only positive consumptions are allowed

→ (iii) Walras law holds because of local nonsatiation of preferences. If we drop local nonsatiation, we can only ensure that $pz(p) \leq 0$ since $z(p)$ is the aggregate excess demand function and all individuals demands are such that every agent's budget constraint is satisfied (although not necessarily with equality). Take:



Agent 1 has a satiation point. When faced with prices $\hat{p}$, agent 1 chooses his satiation point, since it is affordable; agent 2 chooses $\hat{x}^2$.
Since at $\hat{p}$ there is excess supply of both goods and $\hat{p}\, z(\hat{p}) = \underset{(+)}{\hat{p}_1} \underset{(-)}{z_1(\hat{p})} + \underset{(+)}{\hat{p}_2} \underset{(-)}{z_2(\hat{p})} < 0$

So (iii) fails (remember it had to hold for all p).
Also, the offer curves don't intersect and there is no p s.t. $z(p) = 0$
Notice however that $z(p)$ is continuous and bounded below (demand is never negative)

→ (iv) The excess demand is bounded below since we impose non-negative consumptions and finite endowments.
If we allow for negative consumptions and we need the consumption of a certain good to be arbitrarily negative, we must also change the assumptions on the desirability of the goods (i.e. we must drop strong monotonicity).
If good ℓ is actually a bad for some consumer and he can consume negative amounts of that good, he will choose to consume $-\infty$

An alternative approach would be to allow the endowment of some good to be infinite and the consumers to be satiated for that good at some point (the excess supply would therefore be infinite for all prices). Of course, there would be no equilibrium.
One example of an excess demand function that violates (iv) by allowing for negative consumptions of good 2 but still verifies (i) and (iii) is

$$z(p) = \begin{cases} \left(\frac{p_2}{p_1}\omega_2,\ -\omega_2\right) & \text{if } p_1 \leq p_2 \\ \left(\frac{p_1}{p_2}\omega_2,\ -\left(\frac{p_1}{p_2}\right)^2 \omega_2\right) & \text{if } p_1 > p_2 \end{cases}$$

Notice that as $p_1 \to 0$, $z_1(p)$ explodes and as $p_2 \to 0$, $z_1(p)$ explodes and $z_2(p) \to -\infty$

Also $\begin{cases} z_1(p) > 0 \ \ \forall p \geq 0 \text{ and } z_2(p) < 0 \ \ \forall p \\ z_1(p) < 0 \ \ \forall p \leq 0 \end{cases}$ So there is no equilibrium.

→ (ii) states $z(p)$ homogeneous of degree 0. This is not a consequence of the assumptions on preferences and endowments but rather of the structure of the utility maximization problem and of the budget constraint. There is therefore no plausible way of avoiding (ii) unless we consider a completely different model. (In any case, homogeneity of degree 0 was only used in the existence proof in order to normalize prices in the simplex and the only relevant issue if we didn't have H.D.O would be ensuring the price space is closed and bounded i.e. compact and convex.)

3.
a) Efficient allocations are $x_1=0$, y_1 in [0,4]; and $y_1=4$, x_1 in [0,4]. Any interior point of the box allows agent 1 to improve by getting more of y and moving along agent 2's indifference curve.
b) Let the price of x be p and let the price of y be 1. Agent 1 will exhaust income in good y and $y_1=4p$. For agent 2, demands are $x_2=4/p-p/4$ and $y_2=p^2/4$. Market clearing implies $p=0.5(\sqrt{320}-16)$, $x_1=0$, $x_2=4$, $y_1=2(\sqrt{320}-16)$, $y_2=36-2\sqrt{320}$.
c) The core is $x_1=0$, y_1 in]0,4] (since a coalition of person 1 would block $y_1=0$); and $y_1=4$, x_1 in [0,2] (since a coalition of person 2 would block x_1 in]2,4]).

4. The equilibrium may not be efficient; we only need a counterexample:
$U_A=x_Ay_A+x_By_B$
$U_B=x_By_B$
$w_A=(1,0)$
$w_B=(0,1)$
Walrasian equilibrium: $p=1$, $x_A=x_B=y_A=y_B=1/2$, $U_A=1/2$, $U_B=1/4$
(Notice that the walrasian equilibrium does not change with respect to the standard case i.e. it would be the same if $U_A=x_Ay_A$; this is due to the fact that agent A is only choosing x_A and y_A)
Efficient point: $x_B=y_B=1$, $x_A=y_A=0$, $U_A=U_B=1$

5.
- Profit maximization together with constant returns to scale means we will have zero profits in equilibrium; letting p_X denote the price of output X and p_Y denote the price of output Y, and letting w and r respectively denote the prices of labor and capital, this implies that $p_X=p_Y=w+r/2$
- For this Leontieff technology, if input prices are both positive, then profit maximization implies that the quantity of labor equals the amount of X (Y) and the quantity of capital must be equal to X/2 (Y/2); if an input price is zero, then the quantity of that input can exceed this amount.

- In turn, utility maximization for each consumer implies that $p_X.X_i=p_Y.Y_i=0.5(wL_i+rK_i)$.

In equilibrium, it will not be possible to have both input prices be positive, since market clearing for capital would not hold (there would be excess supply). Therefore, the equilibrium price of capital (r) must be zero, and $X=L_x \leq 2K_X$ and $Y=L_Y \leq 2K_Y$, implying that $X+Y=200$. Moreover, from the zero profit conditions, we have $p_X=p_Y=w$ (that can be normalized in the simplex or can all be made equal to 1). Notice that in equilibrium, $L_x=100 \leq 2K_X$ and $L_Y=100 \leq 2K_Y$, with $K_X+K_Y=200$. Each consumer will consume X=100/N and Y=100/N, where N is the number of consumers.

6. General case, just need to plug in the exemple:

Similary, going from Arrow R.E. to A-D

A: $(\hat{x}^1, -, \hat{x}^H, \hat{a}^1, -, \hat{a}^H, \hat{q}_1, -, \hat{q}_S, \hat{p}_1, -, \hat{p}_S)$

A-D: Set $x^{h*} = \hat{x}^h \;\forall h$

$p_s^* = \hat{q}_s \hat{p}_s \;\forall s \qquad p^* = (p_1^*, -, p_S^*)$

Check: (i) b.c. $\quad p^* x^{h*} = \sum_s p_s^* x_s^{h*} = \sum_s \hat{q}_s \hat{p}_s \hat{x}_s^h \underset{\text{by b.c. of A}}{\leq} \sum_s \hat{q}_s (\hat{p}_s w_s^h + \hat{a}^h(s)) =$

$= \sum_s \hat{q}_s \hat{p}_s w_s^h + \underbrace{\sum_s \hat{q}_s \hat{a}^h(s)}_{\leq 0} \leq \sum_s \hat{q}_s \hat{p}_s w_s^h = p^* w^h$ and $p^* x^{h*} \leq p^* w^h$

(again by b.c. of A)

(ii) Consumer optimization

Assume $\exists x^{h\prime}$ better but still satisfies b.c. $\Rightarrow p^* x^{h\prime} \leq p^* w^h$

Then define $a_s^{h\prime} = \hat{p}_s x_s^{h\prime} - \hat{p}_s w_s^h$; if $\hat{q} a^{h\prime} \leq 0$, $x^{h\prime}$ would have been chosen in A as well $\rightarrow$ but indeed,

$\hat{q} a^{h\prime} = \sum_s \hat{q}_s a_s^{h\prime} = \sum_s \hat{q}_s (\hat{p}_s x_s^{h\prime} - \hat{p}_s w_s^h) = p^* x^{h\prime} - p^* w^h \leq 0$

So we reach a contradiction.

(iii) market clearing $\quad \sum_h x_s^{h*} = \sum_h w_s^h \;\forall s$ (direct from Arrow)