Macroeconomics II

– Preliminary –

Nova SBE 2025

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Spring 2025

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Kaldor (1961) stylized facts

On average, over long periods of time:

- The growth rate of output is constant.
- The capital-labor ratio grows at a constant rate.
- The capital-income ratio is constant.
- Capital and labor shares of income are constant.
- Real rates of return are close to constant.
- Dispersion of growth rates across countries.

Growth Theories

- We have seen before that a production technology that combines symmetrically physical and human capital, with constant returns to scale, can generate endogenous (constant perpetual) growth.
- Additionally, we've seen how that model behaves like an economy with only one capital stock, and a linear production function.
- In general, what is required in order to have endogenous (perpetual) growth is a "core" of capital goods whose production does not involve nonreproducible factors.
  - Rebelo (1991)
- Other models have endogenous growth (constant or increasing) through externalities and increasing returns:
  - Romer (1986).

One-sector model

- Recall the one-sector model:

$$K_{t+1} = AK_t^{\alpha} N_t^{1-\alpha} + (1-\delta)K_t - C_t \tag{1}$$

$$\frac{K_{t+1}}{K_t} \equiv g_{Kt} = AK_t^{\alpha-1} N_t^{1-\alpha} + 1 - \delta - \frac{C_t}{K_t}$$

- Even if all resources are devoted to capital accumulation,

$$C_t = 0,$$

  the presence of decreasing returns ($\alpha < 1$) to the only factor of production that can be accumulated, K_t, implies that

$$g_{Kt} \to 0$$

- Only one reproducible factor, K_t,
- Produced with a nonreproducible factor, N_t.

One-sector model: exogenous growth

- The neoclassical model, with constant returns to scale, can be consistent with perpetual growth by making the production function time dependent:

$$K_{t+1} = AK_t^{\alpha}(X_t N_t)^{1-\alpha} + (1-\delta)K_t - C_t \tag{2}$$

- X_t grows at exogenously given rate g_x (technological progress).

- It is possible for output, investment, and consumption to grow at rate g_x.

- Fairly uninteresting growth theory: the growth rate of the economy is determined by a single exogenous aspect of technology.

One-sector model: endogenous growth

- One simple way to generate endogenous growth in a one-sector economy is to assume $\alpha = 1$:

$$K_{t+1} = AK_t + (1-\delta)K_t - C_t \tag{3}$$

- Possible for output, investment, and consumption to grow at constant rates:

$$\frac{K_{t+1}}{K_t} \equiv 1 + g_{Kt} = A + 1 - \delta - \frac{C_t}{K_t}$$

- Indeed, from the Euler equation when $U(C) = C^{\gamma}/\gamma$:

$$C_{t+1}/C_t = g_{ct} = (\beta A)^{1/(1-\gamma)} \tag{4}$$

- Only one reproducible factor, produced without direct or indirect contribution of nonreproducible factors (K_t is the only factor here).

Multi-sector models, endogenous growth

- In general, as long as the technology to produce capital is linearly homogeneous and there is a "core" of capital goods that is produced without nonreproducible factors, perpetual growth is feasible.
- Example: two sectors of production.
- Capital sector uses fraction $1 - \phi_t$ of the available capital stock to produce investment goods I_t with a linear technology:

$$I_t = AK_t(1 - \phi_t) \tag{5}$$

- Capital depreciates at rate δ: $K_{t+1} = (1 - \delta)K_t + I_t$
- Consumption sector combines the remaining capital stock with a nonreproducible factor L to produce consumption goods:

$$C_t = B(K_t\phi_t)^{\alpha}L^{1-\alpha} \tag{6}$$

Multi-sector models, endogenous growth

- Capital can grow at any rate between $(A-\delta)$ and $-\delta$:

$$\phi_t = 0 : g_{kt} = A - \delta,$$
$$\phi_t = 1 : g_{kt} = -\delta.$$

- Consumption can grow at a rate proportional to the growth rate of capital:

$$C_{t+1}/C_t = \left(\frac{K_{t+1}}{K_t}\frac{\phi_{t+1}}{\phi_t}\right)^{\alpha}$$

- Indeed, from the Euler equation with $U(C) = C^{\gamma}/\gamma$:

$$C_{t+1}/C_t = \left[\beta\left(\frac{K_{t+1}}{K_t}\frac{\phi_{t+1}}{\phi_t}\right)^{\alpha-1}\left(A+1-\delta\right)\right]^{1/(1-\gamma)} \quad (7)$$

- Note that B or L do not affect the growth rates.

- Note that $B = A$ and $\alpha = 1$ we are back to the AK model.

Multi-sector models, endogenous growth

- Note that the assumption that capital is linearly produced is not essential.
- The technology to produce capital does not need to be linear but only constant returns to scale: linearly homogeneous.
- Previous class: physical capital and human capital with constant returns technology.
  - Lucas (1988): reproducibility of human capital.
- Reproducible human capital and physical capital, constant returns. Linear production of human capital.
  - Uzawa (1965)

Growth and Externalities

- Arrow (1962), Romer (1986): increasing returns to capital.
- Technology:
$$y_t = f(K_t, k_t, n_t), \tag{8}$$
- Technological progress is external to firms (how k_t affects K_t).
- Given K_t, f is linearly homogeneous in k_t and n_t.
- f increasing in aggregate K_t, $f(\sum k, k, n)$ convex in k.
- K_t aggregate "knowledge", increasing with individual k_t, $K_t = \sum k_t$.
- Romer (1986): Decreasing returns in the production of new knowledge (upper bound on feasible growth rates): Consistent with equilibrium accelerating growth rate.

Romer (1986) example

- Technology: $y_t = A({K_t}^{1-\alpha})k_t^{\alpha}n_t^{1-\alpha}, \quad A > 0, \alpha \in (0,1)$
- CRRA utility: $U(c) = c^{1-\sigma}/(1-\sigma)$, $n_t = 1$.
- Competitive equilibrium:
  - $c_{t+1}/c_t = \big(\beta(r_t + 1 - \delta)\big)^{1/\sigma}$
  - $r_t = F_k = \alpha A K_t^{1-\alpha}k_t^{\alpha-1}n_t^{1-\alpha} \rightarrow \alpha A \quad (K_t = k_t)$
- Social planner:
$$\text{(MPK)} \quad F_k = A \tag{9}$$
- $\alpha < 1$: Higher consumption growth in the planner solution.
- Lower investment in competitive equilibrium outcome.
- What are the private incentives? If all firms choose social optimum k, what's the incentive for one individual firm?

Growth and Externalities

- Uzawa (1965), Lucas (1988): allocation of time affects human capital accumulation, human capital affects productivity of other factors.
- Technology:
$$y_t = f(H_t, k_t, h_t n_t), \tag{10}$$
- Technological progress is external to firms (how h_t affects H_t).
- Given H_t, f is linearly homogeneous in k_t and $h_t n_t$.
- f increasing in aggregate H_t.
- H_t aggregate human capital, increasing with individual h_t.
- As we have seen, no need for external effect of H_t to obtain endogenous growth.
- But this effect implies divergence of social and private value of h_t.

How do taxes affect growth?

Tax Distorted Competitive Equilibrium: Household problem

Consider the competitive equilibrium of an economy where output is produced using physical capital k_t, human capital h_t, and labor n_t. Labor is supplied inelastically and w_t is the wage rate per unit of effective labor. r_t is the rental price of k_t. p_t is the price of time t consumption. A government taxes incomes from physical capital and effective labor, and T_t are lump-sum transfers.
The representative household solves

$$\max \sum_t \beta^t u(c_t) \tag{11}$$

subject to:

$$k_{t+1} \leq (1-\delta_k)k_t + x_{kt}, \tag{12}$$

$$h_{t+1} \leq (1-\delta_h)h_t + x_{ht}, \tag{13}$$

$$\sum_t p_t(c_t + x_{kt} + x_{ht}) \leq \sum_t p_t\left[(1-\tau_{kt})r_t k_t + (1-\tau_{ht})w_t h_t + T_t\right] \tag{14}$$

Tax Distorted Competitive Equilibrium: Firm problem

The firm solves

$$\max p_t \left[F(k_t, z_{2t}) - w_t z_{2t} - r_t k_t \right], \tag{15}$$

where $z_{2t} = n_t h_t$ is the the number of effective labor hours purchased by the firm.

Consumer's budget constraint

We can rewrite the consumer's budget constraint as

$$\sum_t p_t(c_t - T_t) \leq W_0 \equiv \left[(1-\tau_{k0})r_0 + 1 - \delta_k\right]k_0 + \\ \left[(1-\tau_{h0})w_0 + 1 - \delta_h\right]h_0, \qquad (16)$$

where we have use standard no-arbitrage conditions, normalized $p_0 = 1$, and assumed an interior solution $x_{kt} > 0, x_{ht} > 0$ for all t.

Ramsey Problem

The Ramsey Planner's problem is to choose sequences of

$$c_t, k_t, h_t, x_{ht}, x_{kt}, \tau_{kt}, \tau_{ht}, p_t, w_t, \text{ and } r_t$$

to maximize the rep agent welfare subject to the constraints given by the competitive equilibrium conditions.

The initial stocks k_0 and h_0, and the sequences of g_t and T_t are taken as given.

Primal Approach

As we have seen before, the problem can be simplified to eliminate

$$\tau_{kt}, \tau_{ht}, p_t, w_t, \text{ and } r_t$$

using the CE conditions. After doing so, the problem becomes:

$$\max \sum_t \beta^t u(c_t) \tag{17}$$

subject to:

$$\sum_t \beta^t (c_t - T_t) u_c(t) - \mathcal{W}_0 = 0, \tag{18}$$

$$c_t + x_{kt} + x_{ht} + g_t = F(k_t, h_t), \tag{19}$$

$$k_{t+1} \leq (1 - \delta_k) k_t + x_{kt}, \tag{20}$$

$$h_{t+1} \leq (1 - \delta_h) h_t + x_{ht}, \tag{21}$$

$$k_0 \text{ and } h_0 \text{ given.} \tag{22}$$

$\mathcal{W}_0 =$
$u_c(0)\left\{ \left[(1 - \tau_{k0}) F_k(0) + 1 - \delta_k\right] k_0 + \left[(1 - \tau_{h0}) F_h(0) + 1 - \delta_h\right] h_0 \right\}$

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  - So on for $t = 2, 3, ...$
- Under these assumptions, we obtain that

$$\lim_{t\to\infty} \tau_{kt} = 0, \qquad \lim_{t\to\infty} \tau_{ht} = 0. \tag{23}$$

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- Under these assumptions, we obtain that

$$\lim_{t \to \infty} \tau_{kt} = 0, \qquad \lim_{t \to \infty} \tau_{ht} = 0. \tag{23}$$

- Note that this result holds despite the assumption that labor supply is inelastic.

Tax Distorted Competitive Equilibrium: Example

Suppose that consumer preferences are given according to

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

and the production function is

$$F(k,h) = Ak^{\alpha}h^{1-\alpha}, \text{ with } \delta_k = \delta_h = \delta.$$

We can show that if a steady-state balanced growth path exists, it must be that:

$$\frac{h_t}{k_t} = \frac{(1-\alpha)(1-\tau_h)}{\alpha(1-\tau_k)} \tag{24}$$

- A higher τ_h decreases the steady state h/k.
- A higher τ_k increases the steady state h/k.

Tax Distorted Competitive Equilibrium: Example

We can also show that along a balanced growth path equilibrium:

$$\frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = \frac{h_{t+1}}{h_t} = \\ = \left(\beta\big(A\big[(1-\alpha)(1-\tau_h)\big]^{1-\alpha}\big[\alpha(1-\tau_k)\big]^{\alpha} + 1 - \delta\big)\right)^{1/\sigma} \tag{25}$$

- A higher τ_h or τ_k decreases steady state growth of consumption.

Readings

- "On the mechanics of economic development", R. Lucas (1988)
- "Increasing Returns and Long-Run Growth", P. Romer (1986)
- "Long-Run Policy Analysis and Long-Run Growth", S. Rebelo (1991)
- L&S (2018): 15