1. **Problem**
   Assume you have three assets: a risk-free bond, Stock A, and Stock B. The risk-free rate is 2%. Stock A has a mean annualized return of 9% and annualized standard deviation of 14%. Stock B has a mean annualized return of 17% and annualized standard deviation of 18%. Correlation between A and B is -0.2.

   What are the optimal weights invested in the risk-free bond, Stock A, and Stock B that gives you a portfolio standard deviation of 23%?

   When answering this question, you may use the formula for the two-asset tangency portfolio below. Recall this assumes two risky assets (A and B), and the notation r_B^e signifies the return of Asset A in excess of the risk-free rate.

$$w_A = \frac{E(r_A{}^e)\sigma_B{}^2 - E(r_B{}^e)\rho_{A,B}\sigma_A\sigma_B}{E(r_A{}^e)\sigma_B{}^2 + E(r_B{}^e)\sigma_A{}^2 - [E(r_A{}^e) + E(r_B{}^e)]\rho_{A,B}\sigma_A\sigma_B}$$

   Solution

   The Tangency Portfolio has 47.87% in A and 52.13% in B. The expected return of the Tangency Portfolio is 13.17% (μ_T) with a standard deviation of 10.38% (σ_T).

   Now we need to allocate the capital between riskfree asset and risky (tangency portfolio) asset. Allocation point would be on the *capital allocation line* such that $\sigma_p = 23\%$. Thus if w is the weight of optimal portfolio assigned to the risk free asset, then $(1 - w) * \sigma_T = 23\%$. Thus optimal portfolio invests -121.52% in the riskfree asset, $(100\% - (-121.52\%) * 47.87\% =$ 106.05% in A and the remaining 115.47% in B.

2. **Problem**

   Consider the rates of return of stocks X and Y:

Year	r_x	r_y
1	8%	1%
2	8%	2%
3	10%	21%
4	8%	9%
5	-4%	-3%

   (a) Calculate the arithmetic average return on these stocks over the sample period.

   (b) Which stock has greater dispersion around the mean (standard deviation or variance)?

   (c) Calculate the geometric average return of each stock. What do you conclude?

   Solution

   (a) Arithmetic average: $\bar{r}_x = 6\%$ and $\bar{r}_y = 6\%$

   (b) Dispersion: $\sigma_x = 5.66\%$ and $\sigma_y = 9.43\%$ *(Stock Y has greater dispersion)*

   Note: Remember that when calculating variance from historical data, we divide by $N - 1$ when calculating the average.

(c) Geometric mean is given by:

$$GM(r_x) = (1.08 * 1.08 * 1.1 * 1.08 * 0.96)^{1/5} - 1 = 5.87\%$$

$$GM(r_y) = (1.01 * 1.02 * 1.21 * 1.09 * 0.97)^{1/5} - 1 = 5.68\%$$

Despite the fact that the two stocks have the same arithmetic average, the geometric average for Y is less than the geometric average for X. (The reason for this result is the fact that the greater variance of Y drives the geometric average further below the arithmetic average.)

3. **Problem**
If the CAPM is valid, is the following situation possible? Explain briefly.

	Expected Return	Standard Deviation
A	17.1%	32%
B	19.35%	28%

Solution

Since part of the total asset's risk can be diversified away, higher standard deviation does not necessarily imply higher expected returns. Therefore, there is no contradiction with CAPM here.

4. **Problem**
Consider the following investment opportunities and assume the CAPM holds.
(a) The market portfolio (S&P 500 Index fund)
(b) The riskless asset (U.S. 3 month Tbill)
(c) FPL stock
(d) RHAT stock
You are also considering two portfolios
(a) Portfolio A contains \$750 worth of FPL stock and \$100 in RHAT stock
(b) Portfolio B contains \$700 worth of the market portfolio, of which you borrowed \$500 at the riskless rate

Note that the correlation between FPL stock returns and RHAT stock returns is 0.07.

Fill in the blanks in the following table.

Portfolio	E(r)	σ	β
100% of FPL stock	15.55%	43%	____
100% of RHAT stock	____	26%	1
Market Portfolio	12.75%	14%	____
Riskless asset	5.75%	____	____
Portfolio A	____	____	____
Portfolio B	____	____	____

Solution

Portfolio	E(r)	σ	β
100% of FPL stock	15.55%	43%	1.4
100% of RHAT stock	12.75%	26%	1
Market Portfolio	12.75%	14%	1
Riskless asset	5.75%	0%	0
Portfolio A	15.22%	38.28%	1.35
Portfolio B	30.25%	49%	3.5

Remember the following.
(a) Betas are additive (in contrast to standard deviation)
(b) The market beta is one, the riskless asset beta is zero
(c) Market risk premium (MRP) is market expected return less riskless return
(d) To get individual assets (portfolios) expected returns add the MRP multiplied by beta to the riskless rate of return

5. **Problem**

Imagine that you are considering an investment in two assets: the FTSE100 and MSCI World. The FTSE100 has a standard deviation of 23% and the MSCI World has a standard deviation of 20%. The correlation of FTSE100 with MSCI World is -0.19. The return on FTSE100 is $\tilde{r}_{FTSE}$ and the return of the MSCI World is $\tilde{r}_{MSCI}$.

(a) Write the formula for the return of an arbitrary portfolio P with a weight of w in the FTSE100 and (1-w) in the MSCI World in terms of $\tilde{r}_{FTSE}$, $\tilde{r}_{MSCI}$ and w.
(b) Use the result from a) to write the formula for the covariance of portfolio P with the FTSE100 (i.e., cov($\tilde{r}_P$,$\tilde{r}_{FTSE}$)) in terms of w.
(c) Use the result from a) to write the formula for the covariance of portfolio P with the MSCI World (i.e., cov($\tilde{r}_P$,$\tilde{r}_{MSCI}$)) in terms of w.
(d) What must be true of cov($\tilde{r}_P$,$\tilde{r}_{FTSE}$) relative to cov($\tilde{r}_P$,$\tilde{r}_{MSCI}$) if portfolio P is the minimum variance portfolio that is a combination of only the FTSE100 and MSCI World?
(e) Imagine that you are very risk averse and want to find the lowest possible variance portfolio that invests in these two assets. What are the weights on the FTSE100 and MSCI World in the minimum variance portfolio that is a combination of those two investments?

Solution

a) *The formula for portfolio return is given by:*

$$\tilde{r}_P = w * \tilde{r}_{FTSE} + (1 - w) * \tilde{r}_{MSCI}$$

b) *The formula for the covariance of portfolio P with the FTSE100 is given by:*

$$cov(\tilde{r}_P, \tilde{r}_{FTSE}) = cov(w * \tilde{r}_{FTSE} + (1 - w) * \tilde{r}_{MSCI}, \tilde{r}_{FTSE})$$

$$=> cov(\tilde{r}_P, \tilde{r}_{FTSE}) = w * cov(\tilde{r}_{FTSE}, \tilde{r}_{FTSE}) + (1 - w) * cov(\tilde{r}_{MSCI}, \tilde{r}_{FTSE})$$

$$=> cov(\tilde{r}_P, \tilde{r}_{FTSE}) = w * \sigma^2_{FTSE} + (1 - w) * \sigma_{FTSE,MSCI}$$

Remember:

$$cov(X, aY + bZ) = a * cov(X, Y) + b * cov(X, Z)$$

c) *The formula for the covariance of portfolio P with the MSCI World is given by:*

$$cov(\tilde{r}_P, \tilde{r}_{MSCI}) = cov(w * \tilde{r}_{FTSE} + (1 - w) * \tilde{r}_{MSCI}, \tilde{r}_{MSCI})$$

$$=> cov(\tilde{r}_P, \tilde{r}_{FTSE}) = w * cov(\tilde{r}_{FTSE}, \tilde{r}_{MSCI}) + (1-w) * cov(\tilde{r}_{MSCI}, \tilde{r}_{MSCI})$$

$$=> cov(\tilde{r}_P, \tilde{r}_{FTSE}) = w * \sigma_{FTSE,MSCI} + (1-w) * \sigma^2_{MSCI}$$

d) In order to obtain minimum variance portfolio, we must have

$$cov(\tilde{r}_P, \tilde{r}_{FTSE}) = cov(\tilde{r}_P, \tilde{r}_{MSCI})$$

Suppose that they were unequal. Then one could sell the one that co-varies more with the portfolio (i.e., adds more to its risk) and replace it with the low covariance asset. This would reduce portfolio variance. One would adjust weights until each asset's covariance with the portfolio is the same.

This leads us to the concept of marginal variance- the increase in a portfolio's variance following the addition of a small stock position, which is proportional to the covariance of the stock with the portfolio.

Hence:

$$cov(\tilde{r}_P, \tilde{r}_{FTSE}) = cov(\tilde{r}_P, \tilde{r}_{MSCI})$$

$$w * \sigma^2_{FTSE} + (1-w) * \sigma_{FTSE,MSCI} = w * \sigma_{FTSE,MSCI} + (1-w) * \sigma^2_{MSCI}$$

e) Substituting the numbers in the above equation, we get:

$$w * (0.23)^2 + (1-w) * (-0.19) * (0.23) * (0.2) = w * (-0.19) * (0.23) * (0.2) + (1-w) * (0.2)^2$$

So we get

$$w = 0.4416$$

We could also get the same weights by differentiating portfolio variance and setting derivative to zero (using a simply minimization problem). This leads to the following formula:

$$w = \frac{\sigma^2_{MSCI} - \sigma_{FTSE} * \sigma_{MSCI} * \rho_{FTSE,MSCI}}{\sigma^2_{FTSE} + \sigma^2_{MSCI} - 2 * \sigma_{FTSE} * \sigma_{MSCI} * \rho_{FTSE,MSCI}} = 0.4416$$

Note that with both approaches, the weights that we obtain are exactly the same.

<u>Check:</u>

$$cov(\tilde{r}_P, \tilde{r}_{FTSE}) = w * cov(\tilde{r}_{FTSE}, \tilde{r}_{FTSE}) + (1-w) * cov(\tilde{r}_{MSCI}, \tilde{r}_{FTSE})$$

$$cov(\tilde{r}_P, \tilde{r}_{FTSE}) = 0.4416 * (0.23) * (0.23) + 0.5584 * (-0.19) * (0.23) * (0.2) = 0.0185$$

$$cov(\tilde{r}_P, \tilde{r}_{MSCI}) = w * cov(\tilde{r}_{FTSE}, \tilde{r}_{MSCI}) + (1-w) * cov(\tilde{r}_{MSCI}, \tilde{r}_{MSCI}) = 0.0185$$

So both the covariances are same!

6. **Problem**
You estimate the following alphas and factor betas for INTC and AMD using the Fama and French 3-factor model:

$$\tilde{r}_{INTC} - r_f = 0.05 + (0.3)MKT + (-1.2)SMB + (1.5)HML + \epsilon_{INTC}$$

$$\tilde{r}_{AMD} - r_f = -0.02 + (0.1)MKT + (-1)SMB + (0.5)HML + \epsilon_{AMD}$$

Assume your estimates of alphas and betas are accurate.

(a) What are the factor betas on MKT, SMB, and HML in a portfolio that is -95% invested in INTC and 195% invested in AMD?
(b) What is the alpha of this portfolio?

Note that we do not subtract the risk-free rate from these factors because they are already zero-cost portfolios by construction. If you construct your own factors, either make them zero cost (i.e., long-short) as Fama and French did, or subtract the risk-free rate.

Solution

$$\alpha = -0.95 * 0.05 + 1.95 * -0.02 = -0.09$$

$$\beta_{mkt,portfolio} = -0.95 * (0.3) + 1.95 * (0.1) = -0.09$$

$$\beta_{smb,portfolio} = -0.95 * (-1.2) + 1.95 * (-1) = -0.81$$

$$\beta_{hml,portfolio} = -0.95 * (1.5) + 1.95 * (0.5) = -0.45$$

7. **Problem**
You estimate the following alphas and factor betas for INTC and AMD using the Fama and French 3-factor model:

$$\tilde{r}_{INTC} - r_f = -0.04 + (1.5)MKT + (1.1)SMB + (0.9)HML + \epsilon_{INTC}$$

$$\tilde{r}_{AMD} - r_f = -0.06 + (1.2)MKT + (0.9)SMB + (1.1)HML + \epsilon_{AMD}$$

Assume your estimates of the factor betas are accurate and the APT expected return relationship holds.

The factor premiums for MKT, SMB, and HML are 8.5%, 3.3%, and 6.5%, respectively. What is the risk premium of a portfolio invested -85% in INTC and 185% in AMD?

Remember, the factor premiums are the λ's. You don't need to worry about subtracting the risk-free rate from the factor premiums here because they are already zero-cost portfolios, hence premiums, by construction.

Solution
First, we will compute the factor betas of the portfolio.

$$\beta_{mkt,portfolio} = -0.85 * (1.5) + 1.85 * (1.2) = 0.95$$

$$\beta_{smb,portfolio} = -0.85 * (1.1) + 1.85 * (0.9) = 0.73$$

$$\beta_{hml,portfolio} = -0.85 * (0.9) + 1.85 * (1.1) = 1.27$$

Hence the risk premium of the portfolio is computed as,

$$r_{portfolio} - r_f = (0.95) * (8.5\%) + (0.73) * (3.3\%) + (1.27) * (6.5\%) = 18.7\%$$

8. **Problem**
Assume you have three assets: a risk-free bond, Stock A, and Stock B. The risk-free rate is 3%. Stock A has a mean annualized return of 12% and annualized standard deviation of 27%. Stock B has a mean annualized return of 22% and annualized standard deviation of 19%. Correlation between A and B is 0.2.
Construct a portfolio with 30% in A and 70% in B. Call this Portfolio P.

(a) What is the ratio of Portfolio P's risk premium to its variance?
(b) What is the ratio of Stock A's risk premium to its covariance with Portfolio P.
(c) Using only these ratios of risk premium to covariance with Portfolio P, argue whether the tangency portfolio is more or less than 30% invested in A.

Solution
Portfolio mean return is given by:

$$r_P = w_A * r_A + w_B * r_B = 19\%$$

Portfolio variance is given by:

$$(\sigma_P)^2 = (w * \sigma_A)^2 + ((1-w) * \sigma_B)^2 + 2 * w * (1-w) * \rho_{A,B} * \sigma_A * \sigma_B = 2.856\%$$

The ratio of P's risk premium to variance (covariance with itself) is 5.6.

Covariance of A with the portfolio is given by:

$$cov(\tilde{r}_P, \tilde{r}_A) = cov(w * \tilde{r}_A + (1-w) * \tilde{r}_B, \tilde{r}_A) = 2.905\%$$

Hence, the ratio of the risk premium of A to its covariance with P is 3.1. The *return per marginal variance* of A is smaller than Portfolio P, so we should sell a little of Stock A and buy a little of P. If we rebalance in this way, we can create a new portfolio that has a higher return than P while maintaining the same variance.

9. **Problem**

Government Motors has just announced an acquisition of Stark Auto, an electronic vehicle manufacturer. Stark Auto's board has approved the deal commencing 1 year from today, and the deal will go through with certainty. Stark Auto's current share price is £634, and in 1 year when the deal closes, each Stark shareholder will receive 7 shares of Government Motors (currently trading at £71) and £137 cash. Because the deal is certain to go through, the Government Motors and Stark Auto share prices become perfectly positively correlated ($\rho = 1$) after the deal is announced. While the standard deviation of Government Motors is 25%, the Stark Auto is less volatile (σ of 19.6%) because of the cash component of the deal.

(a) Write the formula for the variance of a portfolio with a weight w in Government Motors and a weight (1-w) in Stark Auto.

(b) What weights in Government Motors and Stark Auto will give you a portfolio with zero variance?

Solution

(a) *The formula for portfolio variance is given by:*

$$\sigma_p^2 = w^2 * \sigma_{GM}^2 + (1-w)^2 * \sigma_{SA}^2 + 2 * w * (1-w) * \sigma_{GM} * \sigma_{SA} * \rho_{GM,SA}$$

(b) Variance is always positive; therefore, a portfolio with zero variance is a portfolio with minimum variance.

We know that to obtain a portfolio with minimum variance, covariance of each asset with the portfolio (marginal variances) should be equal.

$$cov(\tilde{r}_{GM}, \tilde{r}_p) = cov(\tilde{r}_{SA}, \tilde{r}_p)$$

$$\rightarrow cov(\tilde{r}_{GM}, w\tilde{r}_{GM} + (1-w)\tilde{r}_{SA}) = cov(\tilde{r}_{SA}, w\tilde{r}_{GM} + (1-w)\tilde{r}_{SA})$$

$$\rightarrow w\sigma_{GM}^2 + (1-w)\sigma_{GM,SA} = (1-w)\sigma_{SA}^2 + w\sigma_{GM,SA}$$

$$\rightarrow w * (0.25)^2 + (1-w) * (0.25) * (0.196) = (1-w) * (0.196)^2 + w * (0.25) * (0.196)$$

Solving the one equation and one unknown for w,

$$w = -3.63$$

$$1 - w = 4.63$$

We could also get the same weights by differentiating portfolio variance and setting derivative to zero. This leads to the following formula for the weight assigned to the first asset in order obtain minimum variance portfolio is:

$$w = \frac{\sigma_{SA}^2 - \sigma_{SA} * \sigma_{GM} * \rho_{GM,SA}}{\sigma_{SA}^2 + \sigma_{GM}^2 - 2 * \sigma_{SA} * \sigma_{GM} * \rho_{GM,SA}}$$

(which comes from minimizing the function σ_p^2 over w). This again gives

$$w = -3.63$$

$$1 - w = 4.63$$

So the portfolio would have -363% allocation in Government Motors and a 463% allocation in Stark Auto.

10. **Problem**
Assume you have three assets: a risk-free bond, Stock A, and Stock B. The risk-free rate is 1%. Stock A has a mean annualized return of 10% and annualized standard deviation of 27%. Stock B has a mean annualized return of 12% and annualized standard deviation of 25%. Correlation between A and B is -0.5. Also, the tangency portfolio has 45.79% in A and 54.21% in B.

    (a) What is the ratio of the tangency portfolio's risk premium to its variance?
    (b) What is the ratio of A's risk premium to its covariance with the tangency portfolio?
    (c) What is the intuition that we get from comparing these ratios?

Solution
Tangency Portfolio mean return is given by:

$$r_P = w_A * r_A + w_B * r_B = 11.08\%$$

Tangency Portfolio variance is given by:

$$(\sigma_P)^2 = (w * \sigma_A)^2 + ((1-w) * \sigma_B)^2 + 2 * w * (1-w) * \rho_{A,B} * \sigma_A * \sigma_B = 1.69\%$$

Covariance of A with the tangency portfolio is given by:

$$cov(\tilde{r}_P, \tilde{r}_A) = cov(w * \tilde{r}_A + (1-w) * \tilde{r}_B, \tilde{r}_A) = 1.508\%$$

To solve for this covariance, remember the properties of random variables: $cov(\alpha X + \beta Y, z) = cov(\alpha X, z) + cov(\beta Y, z)$.
In this case that means

$$cov(w \times r_A + (1-w) r_B, r_A) = cov(w \times r_A, r_A) + cov((1-w) r_B, r_A).$$

Then use other properties of covariance to simplify further to bring the w and $(1-w)$ outside the covariance terms.

Hence, the ratio of the tangency portfolio's risk premium to variance (covariance with itself) is 5.97. The ratio of A's risk premium to covariance with the tangency portfolio is also 5.97.

The intuition is that if they are the same, then you cannot increase return per unit of risk by adding or subtracting A. This is exactly what we should have with the tangency portfolio, because the tangency is the best per unit variance, and a different ratio would imply a profitable rebalancing opportunity.