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1. Consider the following optimal growth model with exogenous technological change. Labor productivity grows at rate $1 + \mu > 1$. The technology is given by:

$$C_t + I_t \leq F(K_t, L_t),\ t = 0, 1, ... \tag{1}$$

$$K_{t+1} \leq (1 - \delta)K_t + I_t,\ t = 0, 1, ... \tag{2}$$

$$L_{t+1} = (1 + \mu)L_t,\ t = 0, 1, ... \tag{3}$$

$$L_0 = 1, K_0 \text{ given,} \tag{4}$$

where C_t is aggregate consumption, I_t investment, K_t is the capital stock in period t, L_t is labor, and δ is capital's depreciation rate. The production function F is strictly increasing, concave, continuously differentiable, and displays constant returns to scale (F homogeneous of degree one). Consumer preferences are given by

$$\sum_{t=0}^{\infty} \beta^t C_t^{\gamma} / \gamma, \quad 0 < \beta < 1, \gamma > 0. \tag{5}$$

a. Define the function f by $f(k_t) = F(K_t, 1)$, where $k_t = (1 + \mu)^{-t} K_t$. Normalize the relevant variables as in k_t, and using the normalized (stationary) variables, write the functional equation that describes the problem of a social planner that maximizes utility subject to the technological constraints.

b. Show that there is a unique stationary solution k^* to the problem in **a.**, with k^* given by:

$$1 = \frac{\hat{\beta}}{1 + \mu} \left[(1 - \delta) + f'(k^*) \right], \tag{6}$$

where $\hat{\beta} = \beta(1 + \mu)^{\gamma}$.

c. Compute the corresponding i^* and c^* when $k_t = k_{t+1} = k^*$. Describe the time series implied by the model for C_t, K_t and I_t, for $t = 0, 1, ...$, if $K_0 = k^*$.

d. Suppose instead that $K_0 < k^*$. Compare the evolution of K_t over time to what is obtained in **c**.

2. Consider the following optimal growth model with human capital accumulation. Assume that labor is the only input in the production function $f(n_t)$, which is strictly increasing, strictly concave, and continuously differentiable, with

$$f(0) = 0, \quad \lim_{n\to 0} f'(n) = \infty, \quad \text{and} \quad \lim_{n\to\infty} f'(n) = 0. \tag{7}$$

The worker can accumulate human capital by reducing the number of hours worked. The number of hours available in period t, h_t, is a decreasing function of the ratio of end-of-period to beginning-of-period human capital k_t, according to $h_t = \phi(k_{t+1}/k_t)$. ϕ is a strictly decreasing, strictly concave, and continuously differentiable function, with $\phi(1-\delta) = 1$ for $\delta \in (0,1)$, and $\phi(1+\lambda) = 0$, for some $\lambda > 0$. That is, human capital depreciates at rate δ if all time is devoted to work ($h_t = 1$), and can grow at most at the rate λ if no time is devoted to work ($h_t = 0$). Given the current level of human capital k_t, next period's human capital k_{t+1} satisfies:

$$(1-\delta)k_t \leq k_{t+1} \leq (1+\lambda)k_t, \quad t = 0, 1, ..., \tag{8}$$

with k_0 given. Effective labor supply in period t is given by $n_t = k_t h_t = k_t\phi(k_{t+1}/k_t)$. Let u denote instantaneous utility of the representative consumer, and assume that

$$u(c) = c^\gamma/\gamma,\ 0 < \gamma < 1.$$

The optimal growth problem can be written as

$$\max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(f(n_t)), \tag{9}$$

subject to (8) and

$$c_t = f(n_t), \quad n_t = k_t\phi(k_{t+1}/k_t), \quad k_0 \text{ given.} \tag{10}$$

a. Write the functional equation associated with the optimal growth problem defined above.

b. Assume that $f(n) = n^\alpha$, with $\alpha \in (0,1)$. Show that in this case the value function v that solves the equation in **a.** has the form $v(k) = Ak^{\alpha\gamma}$, and that the optimal policy is a constant growth rate for human capital $g(k) = \theta k$, for some $\theta > 0$.

3. Exercise 6.1, 6.2, 6.3, 6.4, 6.7 in Ljunqgvist and Sargent (2018).

4. Exercise 13.4, 13.5, 14.3, 14.8 in Ljunqgvist and Sargent (2018).

References:

Ljunqgvist, L. and Sargent T. J. (2018), ”Recursive Macroeconomic Theory”, fourth edition, The MIT Press.