Macroeconomics II

– Preliminary –

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João Brogueira de Sousa

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NOVA SCHOOL OF BUSINESS & ECONOMICS

Dynamic Programming: other applications

In the following slides, we will see two examples of theories that can be modeled as Dynamic Programming problems.

McCall's model of job search

Consider an unemployed worker searching for a job:

- Each period, the worker draws one offer w from a fixed wage distribution $F(W) = \text{Prob}\{w \leq W\}$, with $F(0) = 0, F(\bar{W}) = 1$ for some finite upper bound wage $\bar{W}$.
- The worker can reject the offer, receiving $y_t = b$ units (unemployment compensation and home production), and wait until the next period to draw another offer from F.
- Or he can accept the offer, receiving $y_t = w$ units forever.
- Quitting or firing is not permitted.
- The initially unemployed worker seeks to maximize the expected value of lifetime income:

$$\sum_{t=0}^{\infty} \beta^t y_t, \tag{1}$$

where $0 < \beta < 1$ is a discount factor.

McCall's model of job search

Let $v(w)$ denote the expected value of (1) for an unemployed worker who has the offer w, and is deciding to accept or reject that offer. The value function v satisfies the Bellman equation:

$$v(w) = \max_{\text{accept, reject}} \left\{ \frac{w}{1-\beta}, b + \beta \int_0^B v(w')dF(w') \right\} \qquad (2)$$

- $\frac{w}{1-\beta}$: value of receiving w every period, after accepting the offer.
- $b + \beta \int_0^B v(w')dF(w')$: receiving b one period and discounted expected value of a new offer next period.
- Note that first term is linear in w, second term is a constant.

McCall's model of job search

$$v(w) = \begin{cases} b + \beta \int_0^B v(w') dF(w') = \dfrac{\bar{w}}{1-\beta} & \text{if } w \leq \bar{w} \\ \dfrac{w}{1-\beta} & \text{if } w \geq \bar{w} \end{cases} \tag{3}$$

Reservation wage: $\bar{w}$.
Evaluated at $\bar{w}$, $v(w)$ implies:

$$\frac{\bar{w}}{1-\beta} = b + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta} dF(w') + \beta \int_{\bar{w}}^B \frac{w'}{1-\beta} dF(w') \tag{4}$$

One equation in unknown $\bar{w}$.

McCall's model of job search

The former is equivalent to (Show this):

$$\bar{w} - b = \frac{\beta}{1-\beta} \int_{\bar{w}}^{B} (w' - \bar{w}) dF(w') \quad (5)$$

- LHS: Cost of searching one more period, when the offer is $\bar{w}$.
- RHS: Discounted expected benefit of searching one more period and getting an offer $w' > \bar{w}$.

Define:

$$h(w) = \frac{\beta}{1-\beta} \int_{\bar{w}}^{B} (w' - \bar{w}) dF(w') \quad (6)$$

Then:

$$h'(w) = -\beta/(1-\beta)[1 - F(w)] < 0 \quad (7)$$

$$h''(w) = \beta/(1-\beta)F'(w) > 0 \quad (8)$$

$\implies$ $h(w)$ is convex.

Reservation Wage

Figure: From L&S (2018), Chapter 6.

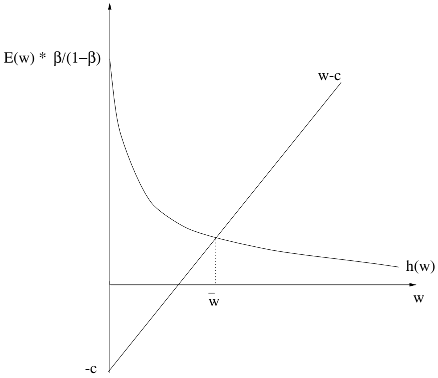

Figure 6.3.2: The reservation wage $\overline{w}$ that satisfies $\overline{w} - c = [\beta/(1-\beta)] \int_{\overline{w}}^{B} (w' - \overline{w}) dF(w') \equiv h(\overline{w})$.

Reservation Wage Properties

1. An increase in the value of the outside option b increases the reservation wage, $\bar{w}$.
2. Given b, a mean-preserving increase in risk increases the reservation $\bar{w}$.

Mean-preserving spread $\tilde{F}$:

$$\int_0^w [\tilde{F}(w') - F(w')]dw' \geq 0, \quad \int_0^B [\tilde{F}(w') - F(w')]dw' = 0. \quad (9)$$

Homework: Show 2.

Unemployment Dynamics

- Probability of rejection: $F(\bar{w})$
- Unemployment duration:

$$Ud = F(\bar{w})(1 - F(\bar{w}) + 2F(\bar{w})^2(1 - F(\bar{w}) + ...$$
$$Ud = F(\bar{w})(1 - F(\bar{w}))^{-1} \quad (10)$$

- Measure of unemployed at t: U_t
- Degenerate unemployment dynamics:

$$U_{t+1} = F(\bar{w})U_t \quad (11)$$

$$U_t \to 0 \quad (12)$$

Asset Pricing

- An asset is a claim on a stream of future payments.
- What is the correct price to pay for such a claim today?
- "Asset Prices in an Exchange Economy", Lucas (1978)

Lucas Tree

- There is an asset (a "Lucas tree") that generates a sequence of perishable consumption goods $\{y_t\}_{t=0}^{\infty}$.
- We assume that this endowment process is Markovian:

$$y_{t+1} = G(y_t, \epsilon_{t+1}) \tag{13}$$

  for some transition function G and shocks ϵ_t i.i.d. with distribution ϕ, and $y_t > 0 \ \forall t$.
- An asset is a claim on all or part of this endowment stream.
- Holding an asset is the only way to transfer wealth (consumption) across periods.

Consumer Preferences

A representative consumer has preferences over consumption sequences:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{14}$$

$\mathbb{E}$ is the expectation operator, $\beta \in (0, 1)$ is a discount factor, and u is a strictly increasing, strictly concave, continuously differentiable utility function.

Asset Markets

The asset is priced $ex - dividend$:

- The seller of the asset retains the period's dividend
- The buyer pays p_t today to purchase a claim on y_{t+1} and the right to sell the claim tomorrow at price p_{t+1}

The share of the asset held by the consumer in period t is denoted by π_t.

Consumer Problem

The consumer maximizes utility (14) subject to the budget constraint:

$$c_t + p_t \pi_{t+1} \leq \pi_t y_t + p_t \pi_t, \ t = 0, 1, ... \tag{15}$$

- The decicion variables at t are: $c_t > 0, \pi_{t+1} \in [0, 1]$.
- The state variables at t are: π_t.
- Competitive market: the consumer takes the prices p_t as given.

Dynamic Programming Problem

- Prices depend on available information. Here, the history of endowments up to period t.
- But we assumed y_t follows a Markov process: only relevant data is current state, y.
- The equilibrium price depends only on y.
- Consumer problem:

$$v(\pi, y) = \max_{c,\pi'} \left\{ u(c) + \beta \int v(\pi', G(y, z))\phi(dz) \right\} \tag{16}$$

subject to

$$c + p(y)\pi' \leq \pi y + p(y)\pi. \tag{17}$$

The solution to this problem is an optimal policy expressing c or π' as a function of the state (π, y).

Dynamic Programming Problem

Using the constraint with equality (u is increasing) to substitute c:

$$v(\pi, y) = \max_{\pi'} \left\{ u[\pi y + p(y)\pi - p(y)\pi'] + \beta \int v(\pi', G(y,z))\phi(dz) \right\} \tag{18}$$

FOC:

$$u'(c)p(y) = \beta \int v_1'(\pi', G(y,z))\phi(dz) \tag{19}$$

v_1' denotes the first derivative of v w.r.t. the first argument.
EC:

$$v_1'(\pi, y) = u'(c)(y + p(y)) \tag{20}$$

$$\implies p(y) = \int \beta \frac{u'(c')}{u'(c)} (G(y,z) + p(G(y,z)))\phi(dz) \tag{21}$$

Solving for Equilibrium Prices

- In equilibrium, it must be that
  - the rep. consumer holds the asset in full $\pi_t = 1$ (no trade), and
  - consumes the endowment $c_t = y_t$.
- Prices are consistent with the above.
- The pricing equation becomes:

$$p(y) = \int \beta \frac{u'(G(y,z))}{u'(y)} (G(y,z) + p(G(y,z)))\phi(dz) \quad (22)$$

- Note that is a functional equation in $p(y)$.
- Sequential problem counterpart:

$$p_t = \mathbb{E}_t \left[\beta \frac{u'(y_{t+1})}{u'(y_t)} (y_{t+1} + p_{t+1}) \right] \quad (23)$$

Solving for Equilibrium Prices

- Lucas (1978): we can obtain the equilibrium price by solving the functional equation.
- Define:

$$f(y) = p(y)u'(y). \tag{24}$$

- (22) can be written as functional equation in f:

$$f(y) = h(y) + \beta \int f[G(y,z)]\phi(dz) \tag{25}$$

- $h(y) = \beta \int u'(G(y,z))G(y,z)\phi(dz)$ does not depend on $p(y)$.
- If we obtain f, we can recover $p(y) = f(y)u'(y)^{-1}$.

Equilibrium Prices as a Fixed Point Problem

- Define the operator T mapping f into Tf as:

$$(Tf)(y) = h(y) + \beta \int f[G(y,z)]\phi(dz) \tag{26}$$

- The solution we seek is $f^* : Tf^* = f^*$.
- We can use fixed point theory to study the conditions under which T has a fixed point.
- Indeed, we can show that:
  a. T has exactly one fixed point, f^*,
  b. For any continuous bounded functions f, $T^n f$ converges uniformly to f^*.
- f^* unique fix point: $p^*(y)$ unique equilibrium price function.

Consumption Based Asset Pricing

- Data on measured aggregate consumption c_t, preference parameters (Coef. RRA), predict prices.
- Equity Premium Puzzle (Mehra & Prescott (1985)): model predicted Equity premium too low / Risk free rate too high, compared to the data.

Readings

- L&S (2018): 6, 13.5-13.7
- "Asset Prices in an Exchange Economy", Lucas (1978)
- Computational example: this quantecon.org lecture