Macroeconomics II

– Preliminary –

Nova SBE 2025

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Spring 2025

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One-sector optimal growth model

$$v(k) = \max_{0 \leq y \leq f(k)} \{u(f(k) - y) + \beta v(y)\} \tag{1}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $u : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions and k is fixed. We assume that:

(U1) $0 < \beta < 1$;

(U2) u is continuous;

(U3) u is strictly increasing;

(U4) u is strictly concave;

(U5) u is continuously differentiable;

(T1) f is continuous;

(T2) $f(0) = 0$, and for some $\bar{k} > 0 : k \leq f(k) \leq \bar{k}$, all $0 \leq k \leq \bar{k}$, and $f(k) < k$, all $k > \bar{k}$;

(T3) f is strictly increasing;

(T4) f is weakly concave;

(T5) f is continuously differentiable.

One-sector optimal growth model

Note that $K = [0, \bar{k}]$ is the set of maintainable capital stocks. If $y \in K$, then u and f are bounded on the relevant domains. Under these assumptions, have the following results:

- Under (U1)-(U3) and (T1)-(T3), there exists a unique, bounded, strictly increasing, continuous function v satisfying (1), and the optimal policy correspondence G is nonempty – see Theorems 4.2-4.7 in S&L (1989);
- Additionally, under (U4) and (T4), v is strictly concave and the optimal policy correspondence is single-valued and continuous – Theorem 4.8 in S&L (1989).

One-sector optimal growth model

Stationary Points

- There is a trivial stationary point at $k = 0$.
  - If $k_0 = 0$, then $f(0) = 0$, the capital stock and output are zero in every subsequent period.
- To determine other stationary points, consider the first-order and envelope conditions for problem (1):

$$u'\left[f(k) - g(k)\right] = \beta v'\left[g(k)\right], \tag{2}$$

$$v'(k) = u'\left[f(k) - g(k)\right] f'(k). \tag{3}$$

- At any stationary point we have that $g(k) = k$. A necessary condition for a stationary point implied by (2)-(3) is:

$$1 = \beta f'(k) \tag{4}$$

One-sector optimal growth model

Stationary Points

Under the stated assumptions on f, there exists a unique positive solution to this equation:

$$k^* = f'^{-1}(1/\beta) \tag{5}$$

Since v is strictly concave, we have that

$$\{v'(k) - v'[g(k)]\}[k - g(k)] \leq 0, \forall k \in (0, \bar{k}] \tag{6}$$

with $=$ iff $g(k) = k$. Substituting $v'(k)$ with the envelope condition, and $v'[g(k)]$ with the FOC, and using the fact that $u'(c) > 0$, conclude that:

$$[f'(k) - 1/\beta][k - g(k)] \leq 0, \forall k \in (0, \bar{k}], \tag{7}$$

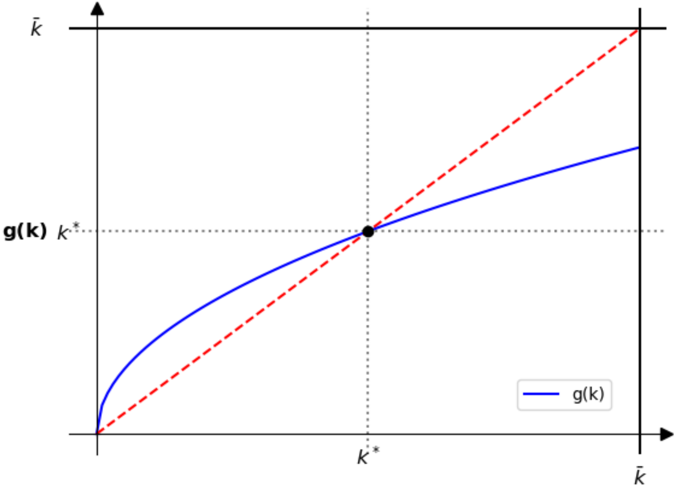
with $=$ iff $g(k) = k$.

- The LHS of (7) is zero at k^*, and holds with strict inequality for $k \neq k^*$.
- Since f is concave, $f'(k) \lesseqgtr 1/\beta$ as $k \gtreqless k^*$.
- It follows that $g(k) \lesseqgtr k$ as $k \gtreqless k^*$.

We have established the following:

Proposition: Let f, u and β satisfy the assumptions above, and let g be the policy function that solves (1). Then g has two stationary points, $k = 0$ and $k^* = f'(1/\beta)$, and for any $k_0 \in (0, \bar{k}]$, the sequence $\{k_t\}_{t=0}^{\infty}$ given by $k_{t+1} = g(k_t)$ converges monotonically to k^*.

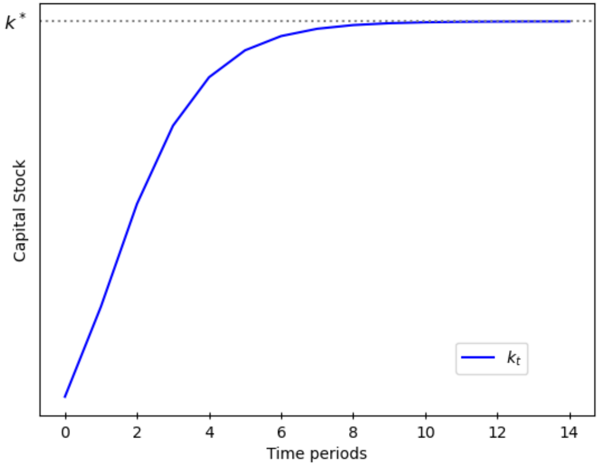
Figure: Policy function, $g(k)$, and steady state capital k^*.



One-sector optimal growth model

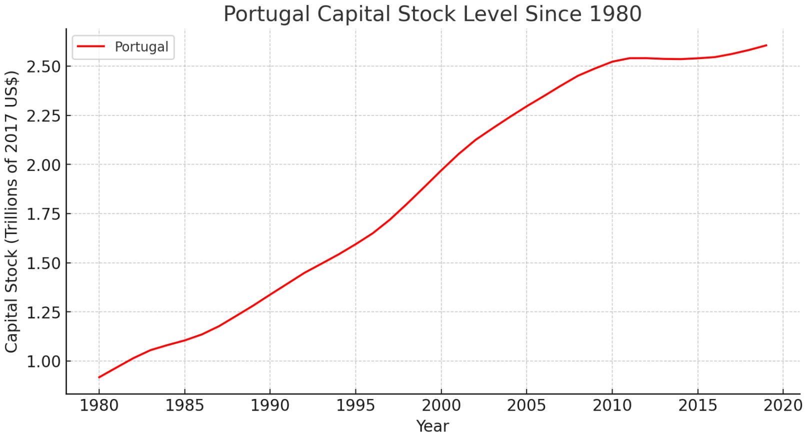
Dynamics

Figure: Capital Stock, k_t, and steady state capital k^*.



Portugal's Capital Stock

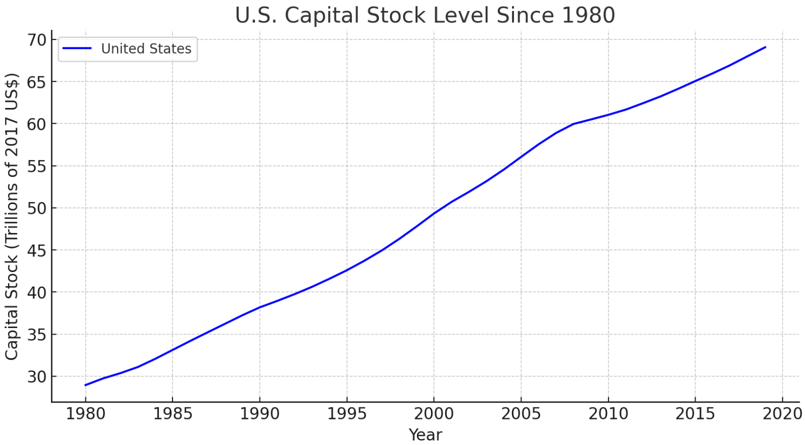
Dynamics



Data: Penn World Table 10.01. Figure: ChatGPT.

U.S.' Capital Stock

Dynamics



Data: Penn World Table 10.01. Figure: ChatGPT.

Adding Growth to the Growth Model

The simplest way to ensure steady-state capital (and consumption) growth is to postulate exogenous technological change (Solow's (1956)), at the constant rate $1 + \mu \geq 1$:

$$L_t = A_t L, \text{ with } A_t = (1 + \mu) A_{t-1}, \tag{8}$$

where L is a fixed stock of labor.
Consumer preferences:

$$\sum_{t=0}^{\infty} \beta^t C_t^{\gamma} / \gamma \tag{9}$$

$0 < \beta < 1$, $\gamma > 0$. Technology:

$$C_t + I_t \leq F(K_t, L_t), t \geq 0 \tag{10}$$

$$K_{t+1} = (1 - \delta) K_t + I_t, t \geq 0 \tag{11}$$

$$L_{t+1} = (1 + \mu) L_t, t \geq 0 \tag{12}$$

$$L_0 > 0, K_0 > 0 \text{ given.} \tag{13}$$

F is strictly increasing, concave, and continuously differentiable

Exogenous Growth Model

Without loss of generality, normalize $L_0 = 1$. Then define:

$$k_t = (1+\mu)^{-t} K_t; \quad c_t = (1+\mu)^{-t} C_t; \quad i_t = (1+\mu)^{-t} I_t, t \geq 0. \tag{14}$$

Additionally, define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(k_t) = F(k_t, 1)$. Using these definitions, we can write the optimal growth model in functional form as:

$$v(k) = \max_{\{c,k'\}} \{c^\gamma/\gamma + \beta(1+\mu)^\gamma v(k')\} \tag{15}$$

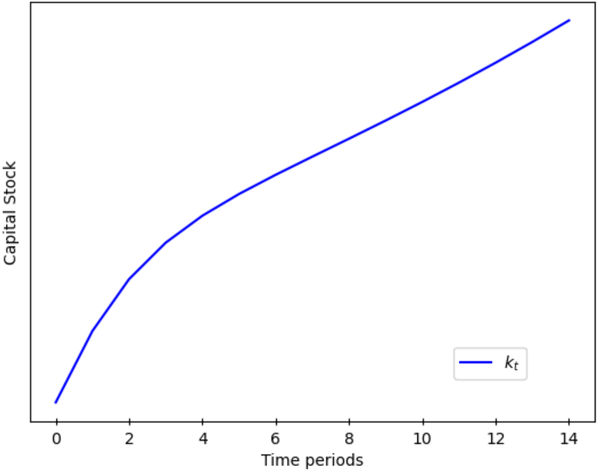
subject to $c + (1+\mu)k' \leq f(k) + (1-\delta)k$, k given.

Previous analysis goes through, provided $\beta(1+\mu)^\gamma < 1$.

Exogenous Growth Model

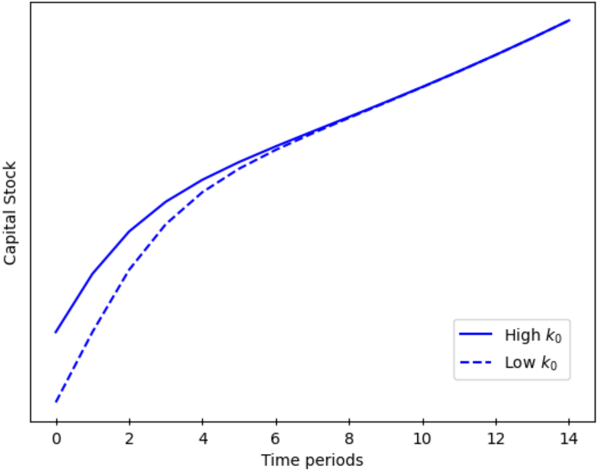
Dynamics

Figure: Capital Stock, k_t, and steady state growth.

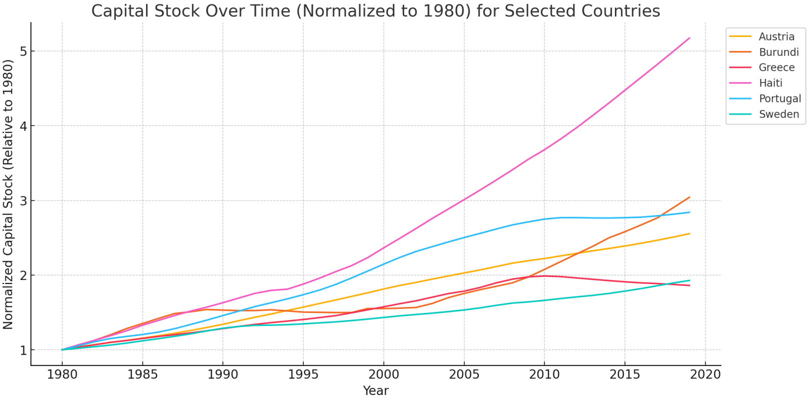


Exogenous Growth Model
Dynamics

Figure: Capital Stock, k_t, from two different k_0.



Growth rates in selected countries



Data: Penn World Table 10.01. Figure: ChatGPT.

Exogenous Growth

- Different countries may have different growth rates:

$$1 + \mu^i, \text{ country } i$$

- Why? And if it depends on country, is it exogenous?
- Some countries converge to the same $(1+\mu)^t f(k^*)$, others pass others in output, over time.
- *Exogenous* movement of the production possibilities frontier over time.
- Change in the production frontier does not require any resources.
- Typically, we think of R&D activities, education, etc. These do take up resources, labor, and capital.

Endogenous Growth

- New theories during the 1990s, starting in U. of Chicago with Paul Romer et al.: endogenous growth (the rate at which technology advances).
- In these models, the technology set is independent of time (and country) – what is possible to do –, but the optimal choices about what to do are different.
- What can be done in a given country and time depends on the level of 'knowledge', and how this 'knowledge' evolves over time is modeled explicitly.
- Increasing 'knowledge' over time requires resources.
- Different assumptions about the form 'knowledge' takes, how it is shared and transmitted across individuals, time, location, etc.

Endogenous Growth Model

Recall the technology in the exogenous growth model seen before:

$$c_t + x_t \leq F(k_t, (1+\mu)^t n_t) \tag{16}$$

Here we endogenize the growth rate of productivity: $(1+\mu)^t$

$$\begin{aligned} c_t + x_{ht} + x_{kt} &\leq F(k_t, h_t n_t), && (17) \\ h_{t+1} &\leq x_{ht} + (1-\delta_h) h_t, && (18) \\ k_{t+1} &\leq x_{kt} + (1-\delta_k) k_t, && (19) \\ h_0, k_0 &\text{ fixed}, && (20) \end{aligned}$$

with h_{t+1} optimally chosen.

Question: under what conditions the optimal path of h_t is $(1+\mu)^t h_0$, and what determines μ?

Endogenous Growth Model

A simple version of the model can easily be expressed as a dynamic programming problem. Suppose labor is inelastically supplied: $n_t = 1, \forall t$.

$$v(h, k) = \max_{c,h',k'} \{u(c) + \beta v(h', k')\} \tag{21}$$

subject to

$$c = F(k, h) - h' - k' - (1 - \delta_h)h - (1 - \delta_k)k \tag{22}$$

FOCs and ECs imply:

$$F_h(h', k') + 1 - \delta_h = F_k(h', k') + 1 - \delta_k \tag{23}$$

Assume F is homogenous of degree one:

$$(23) \implies h'/k' = (h'/k')^* = \text{constant}, \ \forall(h, k). \tag{24}$$

Endogenous Growth Model

Assume F is Cobb-Douglas $F(h,k) = k^{\alpha}h^{1-\alpha}$, $\delta_h = \delta_k$:

$$F_h(h',k') + 1 - \delta_h = F_k(h',k') + 1 - \delta_k \tag{25}$$

$$\implies (1-\alpha)\left(\frac{k'}{h'}\right)^{\alpha} = \alpha\left(\frac{h'}{k'}\right)^{1-\alpha} \implies \frac{h'}{k'} = \frac{1-\alpha}{\alpha} \tag{26}$$

and u is CES:

$$\left(\frac{c'}{c}\right)^{\sigma} = \beta[f'(k') + 1 - \delta_k] \tag{27}$$

Note that:

$$F(k'\frac{1-\alpha}{\alpha}, k') = f(k') = Ak', \text{ with } A = \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}, \tag{28}$$

$$f'(k') = A \implies \frac{c'}{c} = (\beta[A + 1 - \delta_k])^{1/\sigma}. \tag{29}$$

Endogenous steady-state growth depends on preferences and technology.

Growth Models

You may have noticed that the endogenous growth model with human capital accumulation, under the assumptions above, reduces to a one-sector growth model, with a linear technology:

$$\max \sum_{t=0}^{\infty} \beta^t c_t^{1-\sigma}/(1-\sigma) \tag{30}$$

s.t.

$$c_t + (1+a)(k_{t+1} - (1-\delta_k)k_t) \leq Ak_t, \tag{31}$$

with k_0 given, $a = (1-\alpha)/\alpha$. (Assume $h_0 = ak_0$.)

Readings

- L&S (2018): 15
- S&L (1989): 5, 6.1