Exercises Week 4

1. VAR Models

1. Consider the following VAR:

$$
\begin{aligned}
y_t &= (1+\beta)y_{t-1} - \beta\alpha x_{t-1} + \epsilon 1t \\
x_t &= \gamma y_{t-1} + (1-\gamma\alpha)x_{t-1} + \epsilon_{2t}
\end{aligned}
$$

Show that this VAR is non-stationary
Solution:
Write the VAR in companion-form. Let

$$X_t = \begin{pmatrix} y_t \\ x_t \end{pmatrix}, \quad A = \begin{pmatrix} 1+\beta & -\beta\alpha \\ \gamma & 1-\gamma\alpha \end{pmatrix},$$

so that

$$X_t = A\, X_{t-1} + \varepsilon_t.$$

The VAR is covariance-stationary if and only if all eigenvalues of A lie strictly inside the unit circle. The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - (1+\beta) & \beta\alpha \\ -\gamma & \lambda - (1-\gamma\alpha) \end{pmatrix} = (\lambda - 1 - \beta)(\lambda - 1 + \gamma\alpha) + \beta\alpha\,\gamma.$$

Expand:

$$
\begin{aligned}
\det(\lambda I - A) &= \lambda^2 - \big[(1+\beta) + (1-\gamma\alpha)\big]\lambda + (1+\beta)(1-\gamma\alpha) + \beta\alpha\,\gamma \\
&= \lambda^2 - (2+\beta-\gamma\alpha)\,\lambda + \Big[(1+\beta)(1-\gamma\alpha) + \beta\alpha\,\gamma\Big].
\end{aligned}
$$

But

$$(1+\beta)(1-\gamma\alpha) + \beta\alpha\,\gamma = 1 + \beta - \gamma\alpha - \beta\gamma\alpha + \beta\gamma\alpha = 1 + \beta - \gamma\alpha.$$

Hence the polynomial is

$$\lambda^2 - (2+\beta-\gamma\alpha)\,\lambda + (1+\beta-\gamma\alpha) = 0.$$

Observe that

$$(1-\lambda)^2 = \lambda^2 - 2\lambda + 1,$$

so subtracting gives

$$\big[\lambda^2 - (2+\beta-\gamma\alpha)\lambda + (1+\beta-\gamma\alpha)\big] - \big[\lambda^2 - 2\lambda + 1\big] = (-\beta+\gamma\alpha)\,\lambda + (\beta-\gamma\alpha) = (\beta-\gamma\alpha)(1-\lambda).$$

Since β and $\gamma\alpha$ are arbitrary parameters, this implies

$$\det(\lambda I - A) = (1-\lambda)\left(\lambda - d\right) \quad \text{for some } d,$$

i.e. one root is exactly $\lambda = 1$. Therefore A has an eigenvalue on the unit circle, and the VAR is *non-stationary.*

2. Consider the following VAR model:

$$\begin{bmatrix} 1 & c_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} 0.85 & 0.15 \\ -0.25 & 0.65 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \tag{1}$$

where

$$\begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \sim iid\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}\right).$$

a) (a) Is this model in structural or reduced form? Carefully justify your answer.
Solution:
This is the *structural* form, because the contemporaneous coefficient matrix

$$B = \begin{pmatrix} 1 & c_{12} \\ 0 & 1 \end{pmatrix}$$

is not the identity. In structural form one writes $B\,x_t = A_1\,x_{t-1} + \varepsilon_t$, whereas the reduced form has $x_t = \Phi_1\,x_{t-1} + v_t$, which only arises after premultiplying by B^{-1}.

b) (b) Is the model under-, just-, or over-identified? What type of identification scheme is applied? What type of ordering between the two endogenous variables does it imply? Carefully justify your answers.
Solution:
For a 2-variable VAR, B has 2 diagonal entries and 1 off–diagonal entry c_{12}. Imposing the zero restriction on the $(2,1)$ position of B gives exactly

$$\frac{k(k-1)}{2} = \frac{2\cdot 1}{2} = 1$$

restriction, which matches the single unknown c_{12}. Hence the system is *just–identified.* Given the identification restriction, it suggest that x_{2t} is the most exogenous and x_{1t} the least exogenous:

$$\begin{aligned} x_{1t} + c_{12}x_{2t} &= 0.85x_{1,t-1} + 0.15x_{2,t-1} + \epsilon_{1t} \\ x_{2t} &= -0.25x_{1,t-1} + 0.65x_{2,t-1} + \epsilon_{2t} \end{aligned}$$

c) (c) Write the reduced-form of the model in (1). Compute the means, variances, and the covariance of the reduced-form shocks, to be called $[v_{1t}, v_{2t}]'$. Carefully justify your answer and show the details. Note that:

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}.$$

Solution:
Note that residuals are:

$$\begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} = \begin{bmatrix} 1 & -c_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

- $$E[U] = E \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} = \begin{bmatrix} E(\epsilon_{1t}) - c_{12}E(\epsilon_{2t}) \\ E(\epsilon_{2t}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $$\begin{aligned} Var[U] =& E \begin{bmatrix} E(u_{1t} - E(u_{1t}))^2 \\ E(u_{2t} - E(u_{2t}))^2 \end{bmatrix} = \\ & E \begin{bmatrix} E(u_{1t})^2 \\ E(u_{2t})^2 \end{bmatrix} \begin{bmatrix} E(\epsilon_{1t}^2 - 2c_{12}\epsilon_{1t}\epsilon_{2t} + c_{12}^2\epsilon_{2t}^2) \\ E(\epsilon_{2t}^2) \end{bmatrix} = \begin{bmatrix} 1 + c_{12}^2 \\ 1 \end{bmatrix} \end{aligned}$$

- $Cov(u_{1t}, u_{2t}) = -c_{12}\sigma_\epsilon^2 = -c_{12}$

d) (d) Compute the impulse response function (IRF) for both variables to a shock of 0.5 to x_{2t} at impact and step 1.
Solution:
Reduced Form VAR:

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} 1 & -c_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.85 & 0.15 \\ -0.25 & 0.65 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} 1 & -c_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

IRFs:

i) At impact: $$\begin{bmatrix} 1 & -c_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -0.5\,c_{12} \\ 0.5 \end{bmatrix},$$

ii) After 1 period: $$\begin{bmatrix} 1 & -c_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.85 & 0.15 \\ -0.25 & 0.65 \end{bmatrix} \begin{bmatrix} -0.5\,c_{12} \\ 0.5 \end{bmatrix}.$$

.

2. VAR Estimation - Computational Problem

You have recently been recruited as a data analyst at RodNob Fund, a hedge fund specializing in algorithmic trading. Your team is exploring predictive strategies to improve portfolio allocation and is particularly interested in modeling stock market returns using a **Vector Autoregressive (VAR) Model**.

Your primary task is to design a **forecast-based trading strategy**. The investment universe consists of the following stocks:

- 'MSFT': Microsoft
- 'AAPL': Apple
- 'TSLA': Tesla
- 'NFLX': Netflix
- 'META': Meta
- 'AMZN': Amazon
- 'GOOGL': Google

The dataset spans from **January 2000 to September 2023** and consists of daily closing prices obtained from a financial data provider (e.g., Yahoo Finance).

Phase 1: Data Preprocessing and Model Estimation

1. Examine the **stationarity** of each return series by determining its order of integration using appropriate statistical tests (e.g., Augmented Dickey-Fuller test).
2. Estimate a **VAR model** using the most recent **1000 trading days** of data as input.
3. Select the optimal lag length by comparing information criteria and choose the model with the lowest AIC.

Phase 2: Portfolio Allocation Strategy

You are required to construct a **dynamic trading strategy**. For that, you will need to use the previous 1000 days' returns as input for the VAR model, fit up to 20-lag VARs and select the model with the lowest AIC, and using that VAR, forecast the next day's returns. Next, you will construct a Portfolio Allocation Strategy based on the following rules:

- If the model estimation fails for a given day, assume a **neutral position** by assigning zero forecasted returns to all stocks.

- For each stock, **go long**[1] if its forecasted return is **positive**; otherwise, hold no position in that stock for the day.
- Construct an **equally-weighted portfolio** consisting only of the stocks with a positive return forecast.

As the lead analyst on this project, the firm's investment committee has requested a full report detailing your findings. Present an evaluation of the strategy's **cumulative returns** over time, comparing its performance against a simple buy-and-hold strategy.

[1]Going long means buying an asset with the expectation that its price will rise.