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1. Consider the following planning problem:

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u\left(f(k_t) - k_{t+1}\right)$$

subject to:

$$0 \leq k_{t+1} \leq f(k_t),\ t = 0, 1, ..., T \tag{1}$$

$$k_0 > 0 \text{ given.} \tag{2}$$

where the discount factor is $0 < \beta < 1$, and $u : R^+ \to R$ is bounded, continuously differentiable, strictly increasing and strictly concave, with $\lim_{c\to 0} u'(c) = \infty$. The production function f satisfies the following properties:

$$f(0) = 0, \quad f'(k) > 0, \quad \lim_{k\to 0} f'(k) = \infty, \quad \lim_{k\to\infty} f'(k) = 0.$$

a. Show that the solution to the planning problem satisfies the first order and boundary conditions:

$$\beta f'(k_t) u'(f(k_t) - k_{t+1}) = u'(f(k_{t-1}) - k_t), \quad t = 1, 2, ..., T, \tag{3}$$

$$k_{T+1} = 0, \quad k_0 \text{ given.} \tag{4}$$

b. Let $f(k_t) = k_t^\alpha$, $\alpha \in (0, 1)$, and let $u(c_t) = \ln(c_t)$. Show that (3) can be converted into a first-order difference equation in z_t, with $z_t = k_t/k_{t-1}^\alpha$. Plot z_{t+1} against z_t together with the 45° line on the same diagram.

c. Using the boundary condition (4), show that the unique solution is:

$$z_t = \alpha\beta \frac{1 - (\alpha\beta)^{T-t+1}}{1 - (\alpha\beta)^{T-t+2}},\ t = 1, 2, ..., T + 1. \tag{5}$$

d. Check that the path for capital given by

$$k_{t+1} = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}} k_t^\alpha,\ t = 0, 1, ..., T. \tag{6}$$

satisfies (3) and (4).

e. Show that, as $T \to \infty$,

$$k_{t+1} = \alpha\beta k_t^{\alpha},\ t = 0, 1, ... \tag{7}$$

f. Plot k_{t+1} as a function of k_t according to (7), together with the 45° line. Show that there is exactly one positive steady state of the system, i.e. a point $k^* > 0 : k_t = k^* \implies k_{t+1} = k_t$. Show that for any $k_0 > 0$, the sequence $\{k_t\}$ given by (7) converges monotonically to k^* as $t \to \infty$. Can convergence occur in a finite number of periods?

g. Calculate $v(k_0)$ given by:

$$v(k_0) = \sum_{t=0}^{\infty} u(c_t) \tag{8}$$

along the consumption path associated with the optimal path for capital according to (7), and verify that v satisfies

$$v(k) = \max_{0 \leq y \leq f(k)} u(f(k) - y) + \beta v(y). \tag{9}$$

2. Exercise 3.1 in Ljunqgvist and Sargent (2018).

References:

Ljunqgvist, L. and Sargent T. J. (2018), ”Recursive Macroeconomic Theory”, fourth edition, The MIT Press.