Macroeconomics II

– Preliminary –

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Sequential Problem

Recall the sequential problem, $\mathcal{SP}$, of a "social planner" who chooses sequences of consumption and capital:

$$\mathcal{SP}: \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$c_t + k_{t+1} \leq f(k_t) \tag{1}$$

$$k_0 \text{ given.} \tag{2}$$

Indirect utility from k_0

Define $v^* : \mathbb{R}^+ \to \mathbb{R}$ as:

$$v^*(k_0) \equiv \sup_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$c_t + k_{t+1} \leq f(k_t) \tag{3}$$

$$k_0 > 0 \text{ given.} \tag{4}$$

$v^*(k_0)$ is the lifetime utility the consumer gets when he solves $\mathcal{SP}$, starting with an initial capital stock k_0.

Note the sup instead of max: in general there may not be a finite valued solution to $\mathcal{SP}$ (sup is always well defined).

Value Function

- Although the problem we stated in $\mathcal{SP}$ was a choice of infinite sequences $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ at time zero, the problem that the planner in fact faces in period 0 is
  - current consumption c_0, and
  - capital for next period, k_1.
- The rest can wait until $t = 1$.
- If we knew the preferences of the planner over c_0 and k_1, we could simply optimize the choice of these quantities subject to

$$c_0 + k_1 \leq f(k_0). \tag{5}$$

Suppose we know the function $v^*(k_0)$.

Recursive Formulation

- With v^* so defined, $v^*(k_1)$ would give us the value of utility from period 1 that could be obtained with k_1 as beginning of the period capital in $t = 1$.
- Then the problem of the planner in period 0 would be:

$$\max_{c_0, k_1} u(c_0) + \beta v^*(k_1) \tag{6}$$

subject to

$$c_0 + k_1 \leq f(k_0) \tag{7}$$
$$k_0 \text{ given.} \tag{8}$$

- If we know v^*, we can solve this problem and obtain the solution $k_1 = g(k_0)$, $c_0 = f(k_0) - g(k_0)$

Functional Equations

- If we solve the problem above, it follows that $v^*(k_0)$ must satisfy:

$$v^*(k_0) = \max_{c_0, k_1} u(c_0) + \beta v^*(k_1) \text{ , s.t. } c_0 + k_1 \leq f(k_0), \; k_0 \text{ given.} \tag{9}$$

- Also, we could do it for any two subsequent periods, so the time subscript is irrelevant.

$$v^*(k) = \max_{c, k'} u(c) + \beta v^*(k') \text{ , s.t. } c + k' \leq f(k), \; k \text{ given.} \tag{10}$$

- The relevant difference is in k, the capital of the current period (fixed) and k', the capital of the next period (choice variable).
- (10) becomes one equation in one unknown function: v^*. It's called a *functional equation*.
- *Dynamic Programming* deals with dynamic optimization problems expressed in terms of functional equations.

DP Problems

- The previous example is not great motivation for an alternative (recursive) formulation: we have just learned how to solve the sequence problem in the previous lectures.
- However, think how you would extend the analysis if capital productivity in the future were subject to random shocks.
- It makes no sense that the solution continues to be a deterministic sequence of capital; it should depend on the history of shocks.
- This makes the problem very difficult to characterize in sequence form.
- But it is almost unchanged if solved recursively: just find the optimal policy function that depends on current capital and current shock.
  - Optimal policy is contingent on the realization of the shock.
- Obviously, the shocks must have certain properties for us to be able to use this method.

DP Problems

Other examples:

- A worker that faces a job offer today that he can either accept or reject and wait until tomorrow to evaluate a new (random) offer.
- A store manager with a stock of items facing a stochastic demand every day, who has to decide to increase stock at a cost or forgo sales he could have made.
- A financial market with portfolio managers who have to decide whether to keep or to sell a stock at the current price, before learning the dividend it will pay next period.

Recursive Problem

The function v^* satisfies a functional equation, the Bellman Equation:

$$v^*(k) = \sup_{\{c,k'\}} u(c) + \beta v^*(k') \tag{11}$$

subject to:

$$c + k' \leq f(k) \tag{12}$$

$$k \text{ given.} \tag{13}$$

Principle of Optimality

Let $\mathcal{C}(\mathbb{R}^+)$ denote the space of bounded continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$. Define the functional operator T: $\mathcal{C}(\mathbb{R}^+) \to \mathcal{C}(\mathbb{R}^+)$:

$$(Tf)(k) = \sup_{\{c,k'\}} u(c) + \beta f(k') \tag{14}$$

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The Principle of Optimality states the following results:

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(c) If $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ solves $\mathcal{SP}$, then it satisfies:

$$v(k_t) = u(c_t) + \beta v(k_{t+1}), \text{ for } t = 0, 1, ... \tag{16}$$

for $v = v^*$.

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(d) Any sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ that satisfies (16) with $v = v^*$ and a certain boundedness condition, is a solution to $\mathcal{SP}$.

PO: conditions

The main assumption is that for all $k_0 > 0$ and allocations $\tilde{z} \in \mathcal{Z}(k_0)$, where $\mathcal{Z}$ is the set of all feasible allocations:

$$\lim_{n \to \infty} \sum_{t=0}^{n} \beta^t u(\tilde{c}_t) \text{ exists.} \tag{17}$$

For more details see: Theorem 4.2-4.5 in S&L (1989).

If u is bounded, and $\beta \in (0, 1)$, (17) holds.

Note that in common specs (e.g. CRRA) $u : \mathbb{R}^+ \to \mathbb{R}$ is not bounded with the sup norm...

A Contraction Mapping

T satisfies *Blackwell's Sufficient Conditions* for a contraction (see Theorem 3.3 in S&L (1989):

Under certain conditions on f, and assuming u is bounded and continuous, and $\beta \in (0,1)$, the operator T defined above maps $\mathcal{C}(\mathbb{R}^+)$ into itself; has a unique fixed point $v^* \in \mathcal{C}(\mathbb{R}^+)$; and for all $v_0 \in \mathcal{C}(\mathbb{R}^+)$:

$$||T^n v_0 - v^*|| \leq \beta^n ||v_0 - v^*||, \; n = 0, 1, 2, ... \tag{18}$$

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It follows that, starting from any initial function $v_0 \in \mathcal{C}(\mathbb{R}^+)$,

defining v_n as:

$$v_n(k) = (T^n v_0)(k), \tag{19}$$

i.e. applying successively T, n times, we have that:

$$||v_n - v^*|| \rightarrow 0 \ \text{ as } n \rightarrow \infty. \tag{20}$$

Value Function Iteration

The previous properties provide a method to find $v*$:

Pick some initial $\tilde{v}$, apply T over and over again, stop when $||T^n\tilde{v} - T^{n-1}\tilde{v}|| \leq \epsilon$ (small enough).

It's called Value Function Iteration.

Recall that v^* is the unique fixed point of T and, due to (b), it is the value of the solution of the original problem $\mathcal{SP}$.

Policy Functions

In addition, if u is continuous, strictly increasing and strictly concave, and f is continuous, strictly increasing and strictly quasiconcave, then v^* is continuous, strictly increasing, and strictly concave. Because of this, the solution to the problem:

$$\max_{\{c,k\}} u(c, l) + \beta v^*(k') \tag{21}$$

subject to

$$c + k' \leq f(k) \tag{22}$$
$$k \text{ given.} \tag{23}$$

exists and is unique for all k. This solution is what we often call the policy functions of the problem:

$$g_c(k), g_{k'}(k), \tag{24}$$

the optimal decisions for consumption and capital in the state k.

Policy Functions

Recall (c) and (d) above. Let $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ denote the solution to problem $\mathcal{SP}$ with the initial condition $k_0 > 0$. Then

$$c_0^* = g_c(k_0) \tag{25}$$
$$k_1^* = g_{k'}(k_0) \tag{26}$$
$$c_1^* = g_c(g_{k'}(k_0)) \tag{27}$$
$$\ldots$$

Analytical Example

In most cases, we cannot find an analytical solution for v^* and the policy function g, and resort to numerical approximation methods. For some special cases, it is possible to do so.

Assume $f(k) = k^\alpha, \alpha \in (0,1)$ and let $u(c) = \ln(c)$.

$$v(k) = \max_{k'} \ln(k^\alpha - k') + \beta v(k') \tag{28}$$

Take $v_0(k) = 0$.

1. $v_1(k) = \max_{k'} \ln(k^\alpha - k') \implies k' = 0, v_1(k) = \ln(k^\alpha) = \alpha \ln(k)$.

2. $v_2(k) = \max_{k'} \ln(k^\alpha - k') + \beta\alpha \ln(k') \implies$
$k' = \dfrac{\alpha\beta}{1+\alpha\beta} k^\alpha$,
$v_2(k) = \ln\left(\dfrac{1}{1+\alpha\beta}\right) + \alpha\beta \ln\left(\dfrac{\alpha\beta}{1+\alpha\beta}\right) + (\alpha + \alpha^2\beta)\ln(k)$

Analytical Example (cont.)

3. Repeat until realize the sequence follows a geometric sequence that converges to:

$$v(k) = \frac{1}{1-\beta}\left(\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right) + \frac{\alpha}{1-\alpha\beta}\ln(k).$$

4. The associated policy function is:

$$k' = \alpha\beta k^{\alpha}$$

Another solution method: guess and verify that $v(k)$ is of the form:

$$v(k) = A + B\ln(k).$$

Substitute and find A and B.

Analytical Example (cont.)

The first order condition of problem (28) is:

$$-\frac{1}{k^{\alpha}-k'}+\beta v'(k')=0 \tag{29}$$

The envelope condition states that at the solution (i.e. when $k'=g(k)$) the derivative of v satisfies:

$$v'(k)=u'(f(k)-g(k))f'(k)=-\alpha k^{\alpha-1}\frac{1}{k^{\alpha}-g(k)} \tag{30}$$

Together imply:

$$-\frac{1}{k^{\alpha}-g(k)}-\beta\alpha g(k)^{\alpha-1}\frac{1}{g(k)^{\alpha}-g(k')}=0 \tag{31}$$

Verify that $k'=g(k)=\alpha\beta\ln(k)$ satisfies this condition.

Envelope Condition

This is how you can derive the envelope condition.
Consider the previous problem:

$$\max_{k'} \ln(k^{\alpha} - k') + \beta v^*(k'), \ k \text{ fixed.} \tag{32}$$

FOC evaluated at the solution $k' = g(k)$:

$$u'(f(k) - g(k))) = \beta v^{*\prime}(g(k)) \tag{33}$$

Define $w(k) = u(f(k) - g(k)) + \beta v^*(g(k))$. Note that we know $w(k) = v^*(k)$. But let's take the derivate of w w.r.t. k:

$$w'(k) = u'(f(k) - g(k))(f'(k) - g'(k)) + \beta v^{*\prime}(g(k))g'(k). \tag{34}$$

Using the FOC, we can write it as:

$$w'(k) = u'(f(k) - g(k))(f'(k) - g'(k)) + u'(f(k) - g(k)))g'(k) \tag{35}$$

Since $w(k) = v^*(k)$, this implies:

$$v^{*\prime}(k) = u'(f(k) - g(k))f'(k) \tag{36}$$

Readings

- L&S (2018): Chapter 1.
- S&L (1989): Chapters 1, 2, and 4.