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1. Consider the following economy populated by a government and a representative household. There is no uncertainty, and the economy and the representative household and government live forever. The government consumes a constant amount $g_t = g > 0$, $t \geq 0$. The government also sets sequences for of taxes, $\{\tau_{ct}, \tau_{kt}, \tau_{ht}\}_{t=0}^{\infty}$. Here τ_{ct}, τ_{kt} are, respectively, possibly time-varying taxes on consumption and on capital income, and τ_{ht} is a time-varying lump-sum tax. The preferences of the household are ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $\beta \in (0,1)$ and $u(.)$ is strictly concave, increasing, and twice continuously differentiable. The representative household is endowed with one unit of labor each period and does not value leisure. The feasibility condition in the economy is:

$$g_t + c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

where k_t is the stock of capital owned by the household at the beginning of period t and $\delta \in (0,1)$ is a depreciation rate. At time 0, the household faces the budget constraint:

$$\sum_{t=0}^{\infty} q_t[(1+\tau_{ct})c_t + k_{t+1} - (1-\delta)k_t] \leq \sum_{t=0}^{\infty} q_t[(1-\tau_{kt})u_t^k k_t + w_t - \tau_{ht}]$$

where we assume that the household inelastically supplies one unit of labor, w_t is the wage rate of date t labor measured in consumption goods at time t; q_t is the price of date t consumption goods measured in terms of time 0 goods, u_t^k is the rental rate of date t capital measured in consumption goods at time t.

A production firm rents labor and capital. The production function is $f(k)n$, where $f' > 0$, $f'' < 0$. The firm maximizes its value

$$\sum_{t=0}^{\infty} q_t[f(k_t)n_t - w_t n_t - u_t^k k_t n_t]$$

where k_t is the firm's capital-labor ratio and n_t are the units of labor it hires.

The government sets g_t exogenously and must set $\tau_{ct}, \tau_{kt}\tau_{ht}$ to satisfy the budget constraint:

$$\sum_{t=0}^{\infty} q_t(\tau_{ct}c_t + \tau_{kt}u_t^k k_t + \tau_{ht}) = \sum_{t=0}^{\infty} q_t g_t. \tag{1}$$

a. Define a competitive equilibrium.

b. Suppose that historically the government had unlimited access to lump-sum taxes and availed itself of them. Thus, for a long time the economy had $g_t = g > 0$, $\tau_{ct} = \tau_{kt} = 0$. Suppose that this situation had been expected to go on forever. Describe how to find the steady-state capital-labor ratio for this economy.

c. In the economy depicted in **b**, prove that the timing of lump-sum taxes is irrelevant.

d. Let k_0 be the steady value of k_t that you found in part **b**. Let this be the initial value of capital at time $t = 0$ and consider the following experiment. Suddenly and unexpectedly, a court decision rules that lump-sum taxes are illegal and that starting at time $t = 0$, the government must finance expenditures using either the consumption tax τ_{ct} or the capital tax τ_{kt}. The value of g_t remains constant at g. Policy advisor number 1 proposes the following tax policy: find a constant consumption tax that satisfies the budget constraint (1), and impose it from time 0 onward. Compute the value of k_t after the economy converges to a new steady-state, under this policy. Also, get as far as you can in analyzing the transition path from the old steady state to the new one.

e. Policy advisor number 2 proposes the following alternative policy. Instead of imposing the increase in τ_{ct}, he proposes to use the capital tax τ_{kt}. Thus, he/she proposes to set $\tau_{kt} = \tau_k$ for $t \geq 0$ to satisfy the budget constraint (1). Please compute the steady-state level of capital associated with this policy. What can you say about the transition path to the new steady-state k_t under this policy?

f. Someone recommends comparing the two alternative policies – relying completely on the consumption tax, as recommended by policy advisor 1, or completely on the taxation of capital, by adviser 2 –, by comparing the discounted utilities of consumption in steady state, i.e., by comparing

$$\frac{1}{1-\beta}u(\bar{c}) \tag{2}$$

in the two equilibria, where $\bar{c}$ is the steady-state value of consumption. Which of the policy alternatives delivers higher welfare, as measured by (2)?

2. Consider the following economy populated by a government and a representative household. The government consumes $\{g_t\}_{t=0}^{\infty}$, and sets sequences of capital and labor income taxes: $\{\tau_{kt}, \tau_{nt}\}_{t=0}^{\infty}$. The preferences of the household are ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \tag{3}$$

where u is the utility function, $\beta \in (0,1)$, c_t is consumption and n_t is labor. The resource constraint is:

$$c_t + k_{t+1} + g_t = f(k_t, n_t) + (1-\delta)k_t, \tag{4}$$

where k_t is the stock of capital at the beginning of period t and $\delta \in (0,1)$ is a depreciation rate. The production function is $f(k,n)$, constant returns to scale, and $f_k, f_n > 0$ (the marginal products of capital and labor, respectively). A representative firm rents capital and labor to maximizes profits every period, given by:

$$f(k_t, n_t) - r_t k_t - w_t n_t,$$

where k_t is the firm's capital rented from the household at rental price r_t, and n_t are the units of labor it hires, with a wage rate w_t.

The household faces the budget constraints:

$$c_t + k_{t+1} + \frac{b_{t+1}}{R_{t+1}} \leq (1-\tau_{nt})w_t n_t + (1-\tau_{kt})r_t k_t + (1-\delta)k_t + b_t, \ t = 0, 1, ... \tag{5}$$

where b_{t+1} is government debt bought in period t with a price $1/R_{t+1}$, maturing in period $t+1$, with b_0 and k_0 given. In what follows we assume $b_0 = 0$.

For a given sequence of government expenditures g_t, the government's flow of tax revenues, expenditures and debt is:

$$g_t + b_t = \tau_{kt} r_t k_t + \tau_{nt} w_t n_t + \frac{b_{t+1}}{R_{t+1}}, \ t = 0, 1, ... \tag{6}$$

a. Define a competitive equilibrium, formulate the household's and the firm's problem, and derive the competitive equilibrium conditions.

Using the competitive equilibrium conditions obtained above, one can write the household's budget constraint in present value as:

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t (1 - \tau_{nt}) w_t n_t + [1 - \delta + (1 - \tau_{k0}) r_0] k_0, \tag{7}$$

with $q_t = (R_1 R_2 ... R_t)^{-1}$, $t \geq 1$ and $q_0 = 1$.

b. Using the equilibrium conditions obtained in the previous question, show that the household's present value budget constraint given in (7) can be recast in terms of the allocation of consumption and labor, as:

$$\sum_{t=0}^{\infty} \beta^t [u_c(t) c_t + u_n(t) n(t)] = u_c(0) W_0, \tag{8}$$

with $W_0 = [(1 - \delta) + (1 - \tau_{k0}) r_0] k_0$. u_c and u_n denote, respectively, the first derivative of u with respect to consumption and labor.

c. Suppose that τ_{k0} is fixed, and the Ramsey planner takes $\mathcal{W}_0 \equiv u_c(0) W_0$ as given. Formulate the Ramsey problem and take the first order conditions that describe its solution.

d. Assume that preferences are given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t) = \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma} - 1}{1 - \sigma} - \eta n_t^{\psi} \right]$$

Show that the solution to the Ramsey problem, found in **c**, must satisfy:

$$\begin{aligned} \frac{u_c(t)}{\beta u_c(t+1)} &= f_k(k_{t+1}, n_{t+1}) + 1 - \delta, t = 0, 1, ...; \\ \frac{u_n(t)}{u_c(t)} &= \lambda f_n(k_t, n_t), t = 0, 1, ... \end{aligned}$$

where λ is a constant term that depends on preference parameters (σ and ψ) and a (constant) Lagrange multiplier.

e. Comparing the conditions in **d** with the competitive equilibrium conditions derived in **a**, what can you conclude about the optimal tax rates on capital, $\{\tau_{kt+1}\}_{t=0}^{\infty}$ and labor, $\{\tau_{nt}\}_{t=0}^{\infty}$?

f. Assume now that the government sets labor, capital, and also consumption taxes: $\{\tau_{nt}, \tau_{kt}, \tau_{ct}\}_{t=0}^{\infty}$. Take an arbitrary (non-optimal) tax policy that sets $\tau_{ct} = 0$, $\tau_{nt} = \hat{\tau}_n$, $\tau_{kt} = \hat{\tau}_k$ for all $t \geq 0$. Let the equilibrium allocation under this policy be $\{\hat{c}_t, \hat{n}_t, \hat{k}_{t+1}\}_{t=0}^{\infty}$. Show that you can support the same $(\hat{.})$ allocation with a different tax policy, namely one that sets $\tau_{kt} = 0$ for all t and time varying taxes τ_{ct}, τ_{nt} for $t \geq 0$. Find expressions for the time varying taxes τ_{ct}, τ_{nt} as functions of the original $(\hat{.})$ tax policy and the $(\hat{.})$ equilibrium allocation. Note that now the competitive equilibrium conditions are different than in question **a.** due to the presence of consumption taxes.

g. Interpret the time-varying taxes that you computed in part **f.** in terms of the intertemporal distortions on consumption that are in effect induced by a constant (positive) tax on capital.

3. Exercise 11.3.**a.** in Ljunqgvist and Sargent (2018).
4. Exercise 11.10, 11.11, 11.12, 11.13 in Ljunqgvist and Sargent (2018).

References:

Ljunqgvist, L. and Sargent T. J. (2018), "Recursive Macroeconomic Theory", fourth edition, The MIT Press.