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1. In the standard growth model studied in class, assume that the representative firm faces a labor productivity that is proportional to the aggregate ratio of physical capital per worker:
$$X_t = \bar{K}_t L, \text{with } \bar{K}_t = \frac{K_t}{L} \tag{1}$$
Note that the firm takes the productivity as given, i.e. does not take into account the effect of its choices of K_t and X_t on $\bar{K}_t$. In this setting, prove if the competitive equilibrium allocation is Pareto Optimal. Explain your result.

2. Consider the following growth model with two sector. The resource constraint in the goods sector is:
$$C_t + K_{t+1} = K_t^{\alpha}(\phi_t X_t)^{1-\alpha} + (1-\delta)K_t,\ (\alpha, \delta) \in (0,1)^2, \tag{2}$$
$\phi_t X_t$ are the effective units of labor, with $\phi_t \in [0,1]$ being the fraction of time allocated to goods production, and X_t the human capital productivity in period t. The second sector is education. The representative household can allocate a fraction $1-\phi_t$ of time to the production of human capital X_t, which evolves according to:
$$X_{t+1} - X_t = A(1-\phi_t)X_t. \tag{3}$$
Assume that preferences have constant intertemporal substitution of consumption:
$$U(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}. \tag{4}$$
Show that a competitive equilibrium of this economy is consistent with a balanced growth path where consumption, capital and human capital growth a constant rates, and ϕ_t is constant, and determine the growth rate of these quantitites. Show whether the competitive equilibriunm allocation is Pareto Optimal.

3. (Adapted from L&S (2018), Ex. 11.3) An infinitely lived representative household has preferences over a stream of consumption of a single good that are given by:
$$\sum_{t=0}^{\infty} \beta^t U(c_t),\ \beta \in (0,1), \tag{5}$$
where u has the standard properties and β is a discount factor. The technology

is

$$c_t + x_t \leq f(k_t)n_t \tag{6}$$

$$k_{t+1} = (1-\delta)k_t + \varphi_t x_t \tag{7}$$

where for $t \geq 0$:

$$\varphi_t = \begin{cases} 1, & \text{if } t < 4 \\ 2, & \text{if } t \geq 4. \end{cases} \tag{8}$$

Output is $f(k_t)n_t$, $f > 0, f' > 0, f'' < 0$, k_t is capital per unit of labor input and n_t is labor input. The household supplies one unit of labor inelastically. The initial stock of capital k_0 is given and owned by the household. Assume that k_0 is at the optimal steady state value for k assuming φ_t had been equal to 1 forever.

(a) Formulate the planning problem for this economy. Find the formulate for the optimal steady state level of capital. How does a permanent increase in φ affect the steady state levels of consumption, capital and investment?

(b) Formulate an Arrow-Debreu (time zero trades) competitive equilibrium, assuming the household owns the stocks of capital and labor and in each period rents them to the firm, choosing the investment rate each period. Compute the necessary first order conditions for the household and the firm.

(c) What is the connection between the solution of the planning problem and the competitive equilibrium in part b? Link the prices in part c to the corresponding objects in the planning problem, in a.

4. Exercise 11.6 in Ljungqvist and Sargent (2018).

5. Exercise 15.2, 15.5, 15.6 in Ljungqvist and Sargent (2018).

References:

Ljungqvist, L. and Sargent T. J. (2018), "Recursive Macroeconomic Theory", fourth edition, The MIT Press.