Macroeconomics II

– Preliminary –

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Fiscal Policy in a Growth Model

- Main reference:
  - Ljungqvist and Sargent 'Recursive macroeconomic theory' (2018), Chapter 11.
- Framework that allow us to analyze how taxes distort production and consumption (and labor) decisions.
- Simple version of the standard growth model with a government that purchases goods g_t every period, and that finances it with taxes.
- Here we take the government actions as given (exogenous).
  - In the second part of the course, we will study optimal government policies.
- The government is described by a sequence of government purchases $\{g_t\}_{t=0}^{\infty}$, taxes $\{\tau_t\}_{t=0}^{\infty}$ and debt $\{B_t\}_{t=0}^{\infty}$.

A representative household has preferences over streams of a single consumption good c_t given by:

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

- $\beta \in (0,1)$: discount factor.
- U is strictly increasing in c_t, twice continuously differentiable, and strictly concave. Common assumptions:
  - $U(c_t) = \log(c_t)$
  - $U(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}, \ \sigma > 1$

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The technology is:

$$c_t + g_t + x_t = f(k_t) \tag{2}$$
$$k_{t+1} = (1 - \delta)k_t + x_t \tag{3}$$

- Single good produced with $f(k_t)$, used for consumption (c_t), government purchases (g_t), or investment (x_t).
- k_t is the stock of physical capital available for production at t,
- $\delta \in (0, 1)$ is the depreciation rate of capital,
- x_t is gross investment,
- f is a linearly homogeneous production function with positive and decreasing marginal product of capital: $f' > 0,\ f'' < 0$.

The representative household flow of funds in $t = 0, 1, ...$ is:

$$(1+\tau_{ct})c_t + k_{t+1} + \frac{1}{1+r_{t+1}} b_{t+1} \leq b_t + (1-\delta+(1-\tau_{kt})u_t^k)k_t - \tau_{ht} \quad (4)$$

- τ_{ct}: consumption tax,
- k_{t+1}: capital holdings in $t+1$ (k_0 is given),
- b_{t+1}: real public debt in $t+1$ units,
- $1/(1+r_{t+1})$: price of b_{t+1} in units of period t goods,
- δ: capital depreciation rate,
- τ_{kt}: capital income tax,
- u_t^k: return on capital between $t-1$ and t,
- τ_{ht}: lump-sum tax.

Fiscal Policy in a Growth Model

Government Budget Constraint

Every period, the government purchases g_t units of goods and repays outstanding debt B_t, using tax revenues and issuing new debt B_{t+1} at price $\frac{1}{1+r_{t+1}}$:

$$g_t + B_t = \tau_{ct} c_t + \tau_{kt} u_t^k k_t + \tau_{ht} + \frac{1}{1 + r_{t+1}} B_{t+1}, \qquad (5)$$

- g_t: government purchase of goods,
- B_t: debt due in period t,
- $\tau_{ct} c_t$: consumption tax revenue,
- $\tau_{kt} u_t^k k_t$: capital income tax revenue,
- τ_{ht}: lump-sum tax,
- B_{t+1}: one-period debt issued in t, due in $t + 1$.

Fiscal Policy in a Growth Model

Government Budget Constraint

In period 0 and 1:

$$g_0 + B_0 = \tau_{c0} c_0 + \tau_{k0} u_0^k k_0 + \tau_{h0} + \frac{1}{1+r_1} B_1 \quad (6)$$

$$g_1 + B_1 = \tau_{c1} c_1 + \tau_{k1} u_1^k k_1 + \tau_{h1} + \frac{1}{1+r_2} B_2 \quad (7)$$

Adding the two conditions we obtain:

$$g_0 + B_0 = \tau_{c0} c_0 + \tau_{k0} u_0^k k_0 + \tau_{h0} + \\ + \frac{1}{1+r_1} [\tau_{c1} c_1 + \tau_{k1} u_1^k k_1 + \tau_{h1} + \frac{1}{1+r_2} B_2 - g_1] \quad (8)$$

Adding all the conditions for $t = 0, 1, \ldots$:

$$\sum_{t=0}^{\infty} q_t g_t + B_0 = \sum_{t=0}^{\infty} q_t \{\tau_{ct} c_t + \tau_{kt} u_t^k k_t + \tau_{ht}\} + \lim_{T \to \infty} q_T B_T \quad (9)$$

$q_t \equiv \frac{1}{(1+r_1)\ldots(1+r_t)}$ (price of period t good in units of period 0); $q_0 = 1$.

Fiscal Policy in a Growth Model

Representative Firm

The firm rents capital from the household and chooses k_t to maximize:

$$f(k_t) - u_t^k k_t,\ t = 0, 1, ... \tag{10}$$

Recall that u_t^k is the rental cost of capital to the firm.

The household owns capital, makes consumption and investment decisions, rents capital to a representative firm and lends to the government. The representative firm chooses capital in order to maximize profits.

DEFINITION: A competitive equilibrium with taxes is a (i) budget-feasible government policy, a (ii) feasible allocation, and a (iii) price system such that, given the price system and the government policy, the allocation solves (iv) the household's problem and (v) the firm's problem.

Fiscal Policy in a Growth Model

Competitive Equilibrium (cont.)

(i) Given B_0 (initial debt), a budget-feasible government expenditure and tax plan $\{g_t, \tau_{ct}, \tau_{kt}, \tau_{ht}\}_{t=0}^{\infty}$ satisfies:

$$\sum_{t=0}^{\infty} q_t g_t + B_0 = \sum_{t=0}^{\infty} q_t \{\tau_{ct} c_t + \tau_{kt} u_t^k k_t + \tau_{ht}\}, \qquad (11)$$

$$\lim_{T \to \infty} q_T B_T = 0. \qquad (12)$$

(ii) Given k_0, a feasible allocation satisfies:

$$c_t + g_t + k_{t+1} - (1-\delta)k_t = f(k_t),\ t = 0, 1, ... \qquad (13)$$

(iii) A price system $\{q_t, u_t^k\}_{t=0}^{\infty}$,

(iv) Given k_0 and b_0, the household chooses $\{c_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}$ to maximize:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \tag{14}$$

$$(1+\tau_{ct})c_t + k_{t+1} + \frac{1}{1+r_{t+1}} b_{t+1} \leq b_t + (1-\delta+(1-\tau_{kt})u_t^k)k_t - \tau_{ht} \tag{15}$$

(v) The firm chooses k_t to maximize:

$$\Pi_t = f(k_t) - u_t^k k_t, \ t = 0, 1, ... \tag{16}$$

Fiscal Policy in a Growth Model

Consumer's problem

Consumer's consumption problem:

$$\max_{\{c_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \quad s.t. \tag{17}$$

$$(1+\tau_{ct})c_t + k_{t+1} + \frac{1}{1+r_{t+1}} b_{t+1} \leq b_t + (1-\delta+(1-\tau_{kt})u_t^k)k_t - \tau_{ht} \tag{18}$$

Lagrangian, with multipliers μ_t:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t) + \sum_{t=0}^{\infty} \mu_t \Big[b_t + (1-\delta+(1-\tau_{kt})u_t^k)k_t - \tau_{ht} - (1+\tau_{ct})c_t - k_{t+1} - \frac{1}{1+r_{t+1}} b_{t+1} \Big]$$

Fiscal Policy in a Growth Model

Consumer's FOC

Consumer's problem first order conditions:

$$(c_t) \quad \beta^t U'(c_t) - \mu_t(1+\tau_{ct}) = 0 \tag{19}$$

$$(k_{t+1}) \quad -\mu_t + \mu_{t+1}(1-\delta+(1-\tau_{kt+1})u^k_{t+1}) = 0 \tag{20}$$

$$(b_{t+1}) \quad -\frac{\mu_t}{1+r_{t+1}} + \mu_{t+1} = 0 \tag{21}$$

$$(\mu_t) \quad b_t + (1-\delta+(1-\tau_{kt})u^k_t)k_t - \tau_{ht} - (1+\tau_{ct})c_t - k_{t+1} - \frac{1}{1+r_{t+1}}b_{t+1} = 0 \tag{22}$$

Fiscal Policy in a Growth Model

Consumer's FOC

(19) and (20) imply:

$$U'(c_t) = \beta U'(c_{t+1}) \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})}[1-\delta+(1-\tau_{kt+1})u^k_{t+1}],\ t=0,1,... \tag{23}$$

(20) and (21) imply:

$$1 + r_{t+1} = 1 - \delta + (1-\tau_{kt+1})u^k_{t+1},\ t = 0,1,... \tag{24}$$

(22) can be rewritten as:

$$\sum_{t=0}^{\infty} q_t(1+\tau_{ct})c_t + \sum_{t=0}^{\infty} q_t\tau_{ht} + \lim_{T\to\infty} q_T k_{T+1} + \lim_{T\to\infty} \frac{q_T}{1+r_{T+1}} b_{T+1} \leq b_0 + (1-\delta + (1-\tau_{k0})u^k_0)k_0 \tag{25}$$

Fiscal Policy in a Growth Model

Firm's FOC

The firm chooses $\{k_t\}_{t=0}^{\infty}$ to maximize:

$$f(k_t) - u_t^k k_t \tag{26}$$

First order conditions:

$$f'(k_t) - u_t^k = 0, \; t = 0, 1, ... \tag{27}$$

Note: Assumptions on U and f imply FOCs are sufficient for a maximum.

Fiscal Policy in a Growth Model

Equilibrium Conditions

For $t = 0, 1, \ldots$:

$$U'(c_t) - \beta U'(c_{t+1}) \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} [1 - \delta + (1 - \tau_{kt+1}) u^k_{t+1}] = 0 \tag{28}$$

$$\sum_{t=0}^{\infty} q_t [(1+\tau_{ct}) c_t + \tau_{ht}] = b_0 + (1 - \delta + (1 - \tau_{k0}) u^k_0) k_0 \tag{29}$$

$$1 + r_{t+1} = 1 - \delta + (1 - \tau_{kt+1}) u^k_{t+1} \tag{30}$$

$$f'(k_t) - u^k_t = 0 \tag{31}$$

$$\sum_{t=0}^{\infty} q_t \{ g_t - \tau_{ct} c_t - \tau_{kt} u^k_t k_t - \tau_{ht} \} = 0 \tag{32}$$

$$f(k_t) - c_t - g_t - k_{t+1} + (1 - \delta) k_t = 0 \tag{33}$$

Fiscal Policy in a Growth Model

Solving for the equilibrium path

- From (33), with k_0 given, if we knew the optimal consumption c_0, we would know the capital stock next period k_1.
- The Euler equation (23) gives precisely the optimal growth rate of (the marginal utility of) consumption between any period t and $t+1$:

$$\frac{U'(c_t)}{U'(c_{t+1})} = \beta \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})}[1-\delta+(1-\tau_{kt+1})u^k_{t+1}] \qquad (34)$$

- Using (33) to substitute out c_t above:

$$\frac{U'\left(f(k_t)-g_t-k_{t+1}+(1-\delta)k_t\right)}{U'\left(f(k_{t+1})-g_{t+1}-k_{t+2}+(1-\delta)k_{t+1}\right)} = \beta \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})}[1-\delta+(1-\tau_{kt+1})u^k_{t+1}],\ t=0,1,... \qquad (35)$$

Fiscal Policy in a Growth Model

Solving for the equilibrium path

- (35) is a second order difference equation in k_t.
- With two boundary conditions we can solve for the path of capital using (35) alone.
- Assume government policy converges: $\tau_{ct} \to \tau_c$, $\tau_{kt} \to \tau_k$, $g_t \to g$, as $t \to T$.
- Idea: since we know the initial k_0, guess (and verify) that the economy converges to a constant steady state with $k_t = \bar{k}$ as $t \to \infty$.
- $\bar{k}$ gives the second boundary, the steady state value of capital, to which the equilibrium path of capital converges.
- Need to verify if a constant capital stock is consistent with remaining equilibrium conditions (given a constant government policy).

Fiscal Policy in a Growth Model

Computing the equilibrium path with the shooting algorithm

Take a given path for the government policies $\{g_t, \tau_{kt}, \tau_{ct}\}$, that eventually becomes constant:

$$\lim_{t \to T} [g_t \; \tau_{kt} \; \tau_{ct}]' = [g \; \tau_k \; \tau_c]' \tag{36}$$

1. Use (35) to obtain $\bar{k}$ by solving:

$$1 = \beta[1 - \delta + (1 - \tau_k) f'(\bar{k})] \tag{37}$$

2. Select a large time index $S >> T$ and guess an initial consumption rate c_0. Compute $U'(c_0)$ and solve for k_1 using (33).
3. For $t > 0$, use (23) to solve for $U'(c_{t+1})$ and c_{t+1}. Use (33) to compute k_{t+2}.
4. Iterate on step 3 to compute a candidate path $\hat{k}_t, \; t = 1, ..., S$.

Fiscal Policy in a Growth Model

Computing the equilibrium path with the shooting algorithm

5. Compute $\hat{k}_S - \bar{k}$.
6. If $\hat{k}_S > \bar{k}$, raise c_0 and compute a new path $\hat{k}_t,\ t = 1, ..., S$. If $\hat{k}_S < \bar{k}$, lower c_0.
7. Search for a value c_0 in this way, until $\hat{k}_S \approx \bar{k}$.

When we reach $\hat{k}_S \approx \bar{k}$, we have numerically found the equilibrium path for k_t and c_t. We can use the remaining equilibrium conditions to find the other variables, such as r_t, u_t^k, q_t.

We can also find the present value of lump-sum taxes $\sum_{t=0}^{\infty} q_t \tau_{ht}$ that clear the intertemporal government budget, (9).

Note that this equilibrium path depends on the particular path of government policy. There are infinitely many equilibria, index by the combinations of $\{g_t, \tau_{kt}, \tau_{ct}\}$ that satisfy (9).

Before turning to the computer to solve the economy's equilibrium path, it's useful to inspect the equilibrium conditions to learn how different taxes affect equilibrium allocations and prices.

$$U'(c_t) = \beta U'(c_{t+1}) \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} [1 - \delta + (1 - \tau_{kt+1}) u_{t+1}^k] \quad (38)$$

$$\sum_{t=0}^{\infty} q_t [(1+\tau_{ct}) c_t + \tau_{ht}] = b_0 + (1 - \delta + (1 - \tau_{k0}) u_0^k) k_0 \quad (39)$$

$$f'(k_t) - u_t^k = 0 \quad (40)$$

$$1 + r_{t+1} = 1 - \delta + (1 - \tau_{kt+1}) u_{t+1}^k \quad (41)$$

$$\sum_{t=0}^{\infty} q_t \{ g_t - \tau_{ct} c_t - \tau_{kt} u_t^k k_t - \tau_{ht} \} = 0 \quad (42)$$

$$f(k_t) - c_t - g_t - k_{t+1} + (1 - \delta) k_t = 0 \quad (43)$$

Fiscal Policy in a Growth Model

Effects of taxes on the equilibrium

Some terminology:

- **Distorting** taxes: the household can affect the tax payment by altering a decision:
  - Consumption tax: $\tau_{ct} c_t$ by altering c_t;
  - Capital income tax: $\tau_{kt} u_t^k k_t$ by altering k_t.
- **Non-Distorting** taxes: the household cannot affect his payments by changing a decision variable:
  - Lump-sum tax: τ_{ht}.

Fiscal Policy in a Growth Model

Lump-sum taxes and Ricardian equivalence

Suppose that for a given stream of government purchases $\{g_t\}_{t=0}^{\infty}$, $\{\tau_{ct}, \tau_{kt}\}_{t=0}^{\infty} = 0$ and $\{\hat{\tau}_{ht}\}_{t=0}^{\infty}$ is such that:

$$\sum_{t=0}^{\infty} \hat{q}_t[g_t - \hat{\tau}_{ht}] = 0, \tag{44}$$

with $\hat{X} = \{\hat{c}_t, \hat{k}_{t+1}, \hat{u}_t^k, \hat{r}_t, \hat{q}_t\}_{t=0}^{\infty}$ the corresponding equilibrium. Then $\hat{X}$ is not affected by a different sequence of taxes $\{\tilde{\tau}_{ht}\}_{t=0}^{\infty}$, provided that

$$\sum_{t=0}^{\infty} \hat{q}_t[g_t - \tilde{\tau}_{ht}] = 0. \tag{45}$$

Ricardian equivalence: different sequences of lump-sum taxes, with the same present value, do not alter equilibrium allocations. The timing of taxes is irrelevant. Only their present value appears in the budget constraints of the government and the household.

Fiscal Policy in a Growth Model

Constant τ_c is not distorting

If $\tau_{ct} = \tau_c$ for $t = 0, 1, ...$, then:

$$U'(c_t) = \beta U'(c_{t+1}) \frac{(1 + \tau_{ct})}{(1 + \tau_{ct+1})} [1 - \delta + (1 - \tau_{kt+1}) u^k_{t+1}] \quad (46)$$

becomes:

$$U'(c_t) = \beta U'(c_{t+1}) [1 - \delta + (1 - \tau_{kt+1}) u^k_{t+1}] \quad (47)$$

and τ_c shows up only in the budget constraints of government and households.

A constant level of τ_c is not distorting.

Homework: Show under which condition this holds.

Fiscal Policy in a Growth Model

Capital taxation is distorting

Constant levels or a varying capital tax τ_{kt} are distorting:

$$U'(c_t) = \beta U'(c_{t+1})\frac{(1+\tau_{ct})}{(1+\tau_{ct+1})}[1-\delta+(1-\tau_{kt+1})u^k_{t+1}] \quad (48)$$

Homework: Show how a constant τ_k distorts the steady state level of capital.