Unless otherwise stated, the model referred to in the questions in the one described in Week 1 Slides.

1. Define a competitive equilibrium in an economy similar to the one studied in class, but with no production, capital or investment; instead with an endowment of consumption goods received by households every period denoted by e_t^i.

2. Show that if $\left\{k_{xt}^j, n_{xt}^j, x_t^j\right\}_{t=0}^{\infty}$ solves

$$\max \sum_{t=0}^{\infty} \pi_{xt}^j \tag{1}$$

with

$$\pi_{xt}^j = p_{xt}x_t^j - w_t n_{xt}^j - r_t k_{xt}^j \tag{2}$$

subject to:

$$x_t^j \leq F_{xt}^j(k_{xt}^j, n_{xt}^j),\ t = 0, 1, ... \tag{3}$$

non-negativity of all variables; (4)

then, in each period, $(k_{xt}^j, n_{xt}^j, x_t^j)$ solves

$$\max \pi_{xt}^j \tag{5}$$

with

$$\pi_{xt}^j = p_{xt}x_t^j - w_t n_{xt}^j - r_t k_{xt}^j \tag{6}$$

subject to:

$$x_t^j \leq F_{xt}^j(k_{xt}^j, n_{xt}^j), \tag{7}$$

non-negativity of all variables. (8)

3. Suppose that there was just one firm which could allocate its purchases of k and n across all the production functions of all the firms in the economy studied in class. Formulate the profit maximization problem of this firm. Show that it would choose the same total purchases as the individual firms, for any sequence of prices $(w_t, r_t, p_{ct}, p_{xt})$.

4. Show that in a competitive equilibrium allocation, for given prices, in every period

and for every firm, there is no way of choosing an alternative set of inputs that produces the same level of output at a lower total cost.

5. Consider the alternative version of the household maximization problem:

$$\max U^i(\{c_t^i, l_t^i\}_{t=0}^{\infty}) \tag{9}$$

subject to the constraints:

$$\sum_{t=0}^{\infty}(p_{ct}c_t^i + p_{xt}x_t^i + w_t l_t^i) \leq \sum_{t=0}^{\infty}(w_t \bar{n}_t^i + r_t k_t^i + \pi_{xt}^i + \pi_{ct}^i), t = 0, 1, ... \tag{10}$$

$$k_{t+1}^i \leq x_t^i, t = 0, 1, ... \tag{11}$$

$$l_t^i + n_t^i \leq \bar{n}_t^i, t = 0, 1, ... \tag{12}$$

$$k_0^i \text{ given,} \tag{13}$$

$$\text{non-negativity of all variables.} \tag{14}$$

In this version, the household sells all of its hours to the market, and buys back the leisure that she wants. Show that this version gives rise to the same choices of consumption, labor, leisure, investment that the version studied in class.

6. Suppose that (F1) F_{ct}^j, F_{xt}^j are continuous, strictly increasing, strictly quasi-concave, weakly concave, and $F(0) = 0$ for all j (and x, c, and t). Additionally, suppose that (F2) F are constant returns to scale: $F(\lambda k, \lambda n) = \lambda F(k, n)$ for any $\lambda \geq 0$, for all (k, n), for all j and t. Show that in a competitive equilibrium, $\pi_{xt}^j = p_{xt}x_t^j - r_t k_{xt}^j - w_t n_{xt}^j = 0$, and the same for consumption goods firms.

7. Show that if the allocation z and $(p_{ct}, p_{xt}, r_t, w_t)$ is a competitive equilibrium, then z and $(\lambda p_{ct}, \lambda p_{xt}, \lambda r_t, \lambda w_t)$ for any $\lambda > 0$ is also a CE.

8. Show that under assumption (A1) in the slides and (F1) above, if an allocation z

is Pareto Optimal, $z \in \mathcal{PO}$, then it is non-wasteful, i.e. it satisfies:

$$
\begin{aligned}
& c_t^j = F_{ct}^j(k_{ct}^j, n_{ct}^j), \ \forall\, j, \\
& x_t^j = F_{xt}^j(k_{xt}^j, n_{xt}^j), \ \forall\, j, \\
& n_t^i + l_t^i = \bar{n}_t^i, \forall\, i, \\
& k_{t+1}^i = x_t^i, \forall\, i, \\
& \sum_{i=1}^{I} c_t^i = \sum_{j=1}^{J_c} c_t^j, \\
& \sum_{j=1}^{J_x} x_t^j = \sum_{i=1}^{I} x_t^i, \\
& \sum_{j=1}^{J_x} n_{xt}^j, + \sum_{j=1}^{J_c} n_{ct}^j = \sum_{i=1}^{I} n_t^i, \\
& \sum_{j=1}^{J_x} k_{xt}^j, + \sum_{j=1}^{J_c} k_{ct}^j = \sum_{i=1}^{I} k_t^i,
\end{aligned}
$$

for all t.

9. Suppose the solution to the problem:

$$\max_{(u^1,...,u^I)\in\mathcal{U}} U^1 \quad \text{s.t. } u^i \geq \bar{u}, i = 2, ..., I \tag{15}$$

exists for some $\bar{u}$, and denote the solution by $(u^{1*}, ..., u^{I*})$.
Show that $(u^{1*}, ..., u^{I*}) \in \mathcal{PF}$. Additionally, show that if $\mathbf{U}(z) = (u^{1*}, ..., u^{I*})$ for some $z \in \mathcal{Z}$, then $z \in \mathcal{PO}$.

10. Show that under (A1) and (F1), for any $\lambda^i \geq 0$, for $i = 1, ..., I$, there is a unique solution to the problem:

$$\max_{z\mathcal{Z}} \sum_i \lambda^i U^i(z), \tag{16}$$

and that this solution is a $\mathcal{PO}$ allocation.

11. Formulate the problem of a sequential market equilibrium in the model studied in class, with k, x and n. What is the relevant 'no-Ponzi' condition for L_T^i as $T \to \infty$? What is the 'no-arbitrage' condition between interest rates on loans and the rental rate on capital?