Macroeconomics II

– Preliminary –

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Introduction

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- Office hours: Wed. 13:00-14:00 (by appointment)
- Classroom meetings:
  - From February 3rd to May 12th
  - Tuesday (D009): 11:00 - 14:00
- Weekly problem sets + readings
- Grading:
  - Midterm (45%)
  - Final (55%)

Macroeconomics

- Macroeconomics is the study of aggregate dynamics of an economy;
- Also the study of dynamics of the entire distribution of individual economic actors.
- We study it using models.
- A model is an artificial economy used to ask questions.
- Solow (1956):

  "All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive."

Goals

- We want a theory that generates output that can be compared to aggregate or distributional data, like a time series of GDP, employment, or the distribution of consumption, income, or employment across households or firms.
- The theory is not meant to be a *realistic* description of the world, although its predictions should not be counterfactual in any important dimension.
- As a first step, we want a model that has choices of consumption, investment, hours, production that result from the private optimization of resources.

Starters

- Private ownership. Price-taking consumers and firms.
- Consumers: suppliers of factors of production and consumers of final goods.
- Firms: users of factors of production and suppliers of final goods.
  - Investment firms
  - Consumption firms
- Could also include:
  - Government,
  - Banks,
  - etc.

Firms

Two types of firms:

1. Investment firms:
   - denoted by $j = 1, ..., J_x$
   - use capital (k^j_{xt}) and labor (n^j_{xt}) to produce investment goods, x
   - goods go to households (HH) for investment
   - production function: $F^j_{xt}(k^j_{xt}, n^j_{xt})$
2. Consumption firms:
   - denoted by $j = 1, ..., J_c$
   - use capital (k^j_{ct}) and labor (n^j_{ct}) to produce consumption goods, c
   - goods go to HH for consumption
   - production function: $F^j_{ct}(k^j_{ct}, n^j_{ct})$

Firm ownership

- Firms are owned by the households:
  - θ^x_{ij}: consumer i's ownership of investment firm j
  - same for c firms

$$\sum_{i=1}^{I} \theta^x_{ij} = 1, \quad 0 \leq \theta^x_{ij} \leq 1; \tag{1}$$

$$\sum_{i=1}^{I} \theta^c_{ij} = 1, \quad 0 \leq \theta^c_{ij} \leq 1. \tag{2}$$

Households

- Infinitely lived, utility maximizing;
- indexed by $i = 1, ..., I$;
- Decide individual consumption c_t^i, hours of labor n_t^i, leisure l_t^i, investment x_t^i;
- Have endowment of hours $\bar{n}_t^i$, firm ownership, initial capital stock k_0^i.
- Utility function: U^i
  - increasing in c and l
  - consumption and leisure are goods

Competitive (Walrasian) Equilibrium

- A sequence of prices for consumption, investment, capital rental, and wage rate:

$$\{p_{ct}, p_{xt}, r_t, w_t\}, \; t = 0, 1, ... \tag{3}$$

- A sequence of quantities:
  - Households:

$$\left\{c_t^i, x_t^i, k_t^i, n_t^i, l_t^i\right\}, \; t = 0, 1, ..., \; i = 1, ..., I \tag{4}$$

  - Investment firms:

$$\left\{k_{xt}^j, n_{xt}^j, x_t^j\right\}, \; t = 0, 1, ..., \; j = 1, ..., J_x \tag{5}$$

  - Consumption firms:

$$\left\{k_{ct}^j, n_{ct}^j, c_t^j\right\}, \; t = 0, 1, ..., \; j = 1, ..., J_c \tag{6}$$

  - Profits for each household: $\pi_{xt}^i, \pi_{ct}^i, \; i = 1, ..., I$.

CE: Households

Such that, for all i, $\left\{c_t^i, x_t^i, k_t^i, n_t^i, l_t^i\right\}_{t=0}^{\infty}$ solves:

$$\max U^i(\left\{c_t^i, l_t^i\right\}_{t=0}^{\infty}) \tag{7}$$

subject to the constraints:

$$\sum_{t=0}^{\infty}(p_{ct}c_t^i + p_{xt}x_t^i) \leq \sum_{t=0}^{\infty}(w_t n_t^i + r_t k_t^i + \pi_{xt}^i + \pi_{ct}^i), t = 0, 1, ... \tag{8}$$

$$k_{t+1}^i \leq x_t^i, t = 0, 1, ... \tag{9}$$

$$l_t^i + n_t^i \leq \bar{n}_t^i, t = 0, 1, ... \tag{10}$$

$$k_0^i \text{ given,} \tag{11}$$

$$\text{non-negativity of all variables.} \tag{12}$$

CE: Investment Firms

Such that, for all $j = 1, \dots, J_x$, $\left\{ k^j_{xt}, n^j_{xt}, x^j_t \right\}_{t=0}^{\infty}$ solves:

$$\max \sum_{t=0}^{\infty} \pi^j_{xt} \tag{13}$$

with

$$\pi^j_{xt} = p_{xt} x^j_t - w_t n^j_{xt} - r_t k^j_{xt} \tag{14}$$

subject to:

$$x^j_t \leq F^j_{xt}(k^j_{xt}, n^j_{xt}), \; t = 0, 1, \dots \tag{15}$$

non-negativity of all variables. (16)

CE: Consumption Firms

Such that, for all $j = 1, ..., J_c$, $\left\{k_{ct}^j, n_{ct}^j, c_t^j\right\}_{t=0}^{\infty}$ solves:

$$\max \sum_{t=0}^{\infty} \pi_{ct}^j \tag{17}$$

with

$$\pi_{ct}^j = p_{ct} c_t^j - w_t n_{ct}^j - r_t k_{ct}^j \tag{18}$$

subject to:

$$c_t^j \leq F_{ct}^j(k_{ct}^j, n_{ct}^j), \ t = 0, 1, ... \tag{19}$$

non-negativity of all variables. (20)

CE: Supply and Demand

Such that:

$$\sum_{i=1}^{I} c_t^i \leq \sum_{j=1}^{J_c} c_t^j,\ t = 0, 1, ... \tag{21}$$

$$\sum_{i=1}^{I} x_t^i \leq \sum_{j=1}^{J_x} x_t^j,\ t = 0, 1, ... \tag{22}$$

$$\sum_{j=1}^{J_x} n_{xt}^j, + \sum_{j=1}^{J_c} n_{ct}^j \leq \sum_{i=1}^{I} n_t^i,\ t = 0, 1, ... \tag{23}$$

$$\sum_{j=1}^{J_x} k_{xt}^j, + \sum_{j=1}^{J_c} k_{ct}^j \leq \sum_{i=1}^{I} k_t^i,\ t = 0, 1, ... \tag{24}$$

CE: Profit Accounting

With:

$$\pi^i_{xt} = \sum_{j=1}^{J_x} \theta^x_{ij} \pi^j_{xt} \ , \forall i, t = 0, 1, ... \tag{25}$$

$$\pi^i_{ct} = \sum_{j=1}^{J_c} \theta^c_{ij} \pi^j_{ct} \ , \forall i, t = 0, 1, ... \tag{26}$$

Competitive Equilibrium: Existence

- Theory of general equilibrium: Walras (1874).
- Conditions for the existence of equilibrium:
  - N goods: Arrow and Debreu (1954), using a fixed point argument (Nash (1951));
  - ∞-many goods: Bewley (1972), using three different arguments (in the paper, it's the "simplest of the three"!!)

Definition: Allocation

- *Allocation*: An allocation is a vector of quantities:

$$z = \left\{ \{ c_t^i, x_t^i, k_t^i, n_t^i, l_t^i \}_{i=1}^{I}, \{ x_{xt}^j, k_{xt}^j, n_{xt}^j \}_{j=1}^{J_x}, \{ c_t^j, k_{ct}^j, n_{ct}^j \}_{j=1}^{J_c} \right\}_{t=0}^{\infty}$$

Definition: Feasible Allocation

- *Feasible allocation*: An allocation is feasible if, for $t = 0, 1, ...$:

$$c_t^j \leq F_{ct}^j(k_{ct}^j, n_{ct}^j),\ x_t^j \leq F_{xt}^j(k_{xt}^j, n_{xt}^j), \forall\ j,$$
$$n_t^i + l_t^i \leq \bar{n}_t^i, \forall\ i,$$
$$k_{t+1}^i \leq x_t^i, \forall\ i,$$
$$\sum_{i=1}^{I} c_t^i \leq \sum_{j=1}^{J_c} c_t^j,\ \sum_{j=1}^{J_x} x_t^j \leq \sum_{i=1}^{I} x_t^i,$$
$$\sum_{j=1}^{J_x} n_{xt}^j, + \sum_{j=1}^{J_c} n_{ct}^j \leq \sum_{i=1}^{I} n_t^i,$$
$$\sum_{j=1}^{J_x} k_{xt}^j, + \sum_{j=1}^{J_c} k_{ct}^j \leq \sum_{i=1}^{I} k_t^i,$$

$+$ non-negativity of all variables.

Notation

Define the set of feasible allocations $\mathcal{Z}$:

$$\mathcal{Z} = \{z | z \text{ is } \textit{feasible}\}$$

Define the function $\mathbf{U} : \mathcal{Z} \to \mathbb{R}^I$ by:

$$\mathbf{U}(z) = \{U^i(\{c_t^i, l_t^i\}_{t=0}^{\infty})\}_{i=1}^{I}$$

Define $\mathcal{U} = \mathbf{U}(\mathcal{Z})$, i.e. the range of $\mathbf{U}$, the Utility Possibility Set. This is the set of all the *feasible* utility vectors.

Definition: Pareto Frontier and Optimal Allocations

The *Pareto Frontier* is the upper contour of the Utility Possibility Set:

$$\mathcal{PF} = \{u \in \mathcal{U} | \nexists \hat{u} \in \mathcal{U} \ s.t. \ ; \hat{u}^i \geq u^i, \forall i \text{ and } \hat{u}^i > u^i \text{ some } i\}.$$

The set of *Pareto Optimal Allocations* is:

$$\mathcal{PO} = \{z \in \mathcal{Z} | U(z) \in \mathcal{PF}\}.$$

$\rightarrow$ If z is a Pareto Optimal Allocation, the corresponding utility $U(z)$ is in the upper contour of the Utility Possibility Set.

Competitive Equilibrium and Pareto Optimality

- We are now in a position to characterize Competitive Equilibrium allocations on welfare grounds.
- Under some conditions, (1) CE allocations are Pareto Optimal, and (2) Pareto Optimal allocations can be "decentralized" (achieved) through a Competitive Equilibrium.
- These two results are the First and Second Theorems of Welfare Economics.
- Some assumptions for what follows:
  - (A1) U is continuous, strictly increasing and strictly concave.
  - (A2) Endowments are positive: $k_0^i > 0$ and $\bar{n}_t^i > 0, \forall t$.

First Welfare Theorem

Theorem 1. Under A1 and A2, if z^* is a Competitive Equilibrium allocation under prices $p^* = \{p^*_{ct}, p^*_{xt}, r^*_t, w^*_t\}_{t=0}^{\infty}$, then z^* is Pareto Optimal.

Sketch of Proof:

1. Suppose not. Then, we can make at least one consumer better off, without harming anyone.
2. The only way to make at least one consumer better off is to increase her wealth, measured at prices p^*, without decreasing it for anyone else.
3. Then we would have to increase *aggregate* wealth.
4. This cannot be feasible at prices p^*.

First Welfare Theorem: Proof

Let's write the HH budget constraint as:

$$\sum_{t=0}^{\infty}(p^*_{ct}c^i_t + w^*_t l_t) \leq \sum_{t=0}^{\infty}(w^*_t \bar{n}^i_t + r_t k^i_t - p^*_{xt}x^i_t + \pi^i_{xt} + \pi^i_{ct}) \quad (27)$$

Note: replaced n_t and rearranged terms.

Now, suppose that this is not true. That is, suppose that there is a feasible allocation $\hat{z}$ that Pareto dominates z^*. Without loss of generality, assume that household 1 is strictly better off with $\hat{z}$.

Since $\{\hat{c}^1_t, \hat{x}^1_t, \hat{k}^1_t, \hat{n}^1_t, \hat{l}^1_t\}$ improves 1's Utility, it must not have been affordable at prices p^* (otherwise 1 would have preferred it):

$$\sum_{t=0}^{\infty}(p^*_{ct}\hat{c}^1_t + w^*_t \hat{l}^1_t) > \sum_{t=0}^{\infty}(w^*_t \bar{n}^1_t + r_t \hat{k}^1_t - p^*_{xt}\hat{x}^1_t + \pi^1_{xt} + \pi^1_{ct}) \quad (28)$$

First Welfare Theorem: Proof

Similarly, since $\hat{z}$ does not make anyone else worse off, it must be that:

$$\sum_{t=0}^{\infty}(p_{ct}^{*}\hat{c}_{t}^{i}+w_{t}^{*}\hat{l}_{t}^{i}) \geq \sum_{t=0}^{\infty}(w_{t}^{*}\bar{n}_{t}^{i}+r_{t}\hat{k}_{t}^{i}-p_{xt}^{*}\hat{x}_{t}^{i}+\pi_{xt}^{i}+\pi_{ct}^{i}), i=2,...,I \quad (29)$$

Summing over all i, these imply:

$$\sum_{i=1}^{I}\sum_{t=0}^{\infty}(p_{ct}^{*}\hat{c}_{t}^{i}+w_{t}^{*}\hat{l}_{t}^{i}) > \sum_{i=1}^{I}\sum_{t=0}^{\infty}(w_{t}^{*}\bar{n}_{t}^{i}+r_{t}\hat{k}_{t}^{i}-p_{xt}^{*}\hat{x}_{t}^{i}+\pi_{xt}^{i}+\pi_{ct}^{i}). \quad (30)$$

Additionally, since z^{*} is a CE, profits under $\hat{z}$ cannot be higher:

$$\sum_{t=0}^{\infty}\hat{\pi}_{ct}^{i} \leq \sum_{t=0}^{\infty}\pi_{ct}^{i},\ \sum_{t=0}^{\infty}\hat{\pi}_{xt}^{i} \leq \sum_{t=0}^{\infty}\pi_{xt}^{i}, i=1,...,I \quad (31)$$

because firms maximize profits at the z^{*} plan.

First Welfare Theorem: Proof

Therefore, we can write:

$$\sum_{i=1}^{I}\sum_{t=0}^{\infty}(p^*_{ct}\hat{c}^i_t + w^*_t \hat{l}^i_t) > \sum_{i=1}^{I}\sum_{t=0}^{\infty}(w^*_t \bar{n}^i_t + r_t \hat{k}^i_t - p^*_{xt}\hat{x}^i_t + \hat{\pi}^i_{xt} + \hat{\pi}^i_{ct}). \tag{32}$$

Now we need to show that this cannot be feasible (resources).

First Welfare Theorem: Proof

Since $\hat{z}$ is feasible by assumption, it must be that, for $t = 0, 1, ...$:

$$\sum_{i=1}^{I} \hat{c}_t^i \leq \sum_{j=1}^{J_c} \hat{c}_t^j \leq \sum_{j=1}^{J_c} F_{ct}^j(\hat{k}_{ct}^j, \hat{n}_{ct}^j), \tag{33}$$

$$\sum_{i=1}^{I} \hat{x}_t^i \leq \sum_{j=1}^{J_c} \hat{x}_t^j, \tag{34}$$

$$\sum_{j=1}^{J_c} \hat{k}_{ct}^j + \sum_{j=1}^{J_x} \hat{k}_{xt}^j \leq \sum_{i=1}^{I} \hat{k}_t^i, \tag{35}$$

$$\sum_{j=1}^{J_c} \hat{n}_{ct}^j + \sum_{j=1}^{J_x} \hat{n}_{xt}^j \leq \sum_{i=1}^{I} \hat{n}_t^i, \tag{36}$$

$$\sum_{i=1}^{I} (\hat{n}_t^i + \hat{l}_t^i) \leq \sum_{i=1}^{I} \bar{n}_t^i, \tag{37}$$

With monotonicity of U^i, we can replace $\leq$ with $=$. Why?

First Welfare Theorem: Proof

Multiplying by the appropriate prices, summing across sectors and t:

$$\sum_{t=0}^{\infty}\sum_{i=1}^{I}(p_{ct}^{*}\hat{c}_t^i + w_t^{*}\hat{l}_t^i) = \sum_{t=0}^{\infty}\sum_{i=1}^{I} w_t^{*}\bar{n}_t^i$$
$$+ \sum_{t=0}^{\infty} p_{ct}^{*}\sum_{j=1}^{J_c} F_{ct}^j(\hat{k}_{ct}^j, \hat{n}_{ct}^j) - \sum_{t=0}^{\infty}\sum_{j=1}^{J_c} r_t^{*}\hat{k}_{ct}^j - \sum_{t=0}^{\infty}\sum_{j=1}^{J_c} w_t^{*}\hat{n}_{ct}^j$$
$$+ \sum_{t=0}^{\infty} p_{xt}^{*}\sum_{j=1}^{J_x} F_{xt}^j(\hat{k}_{xt}^j, \hat{n}_{xt}^j) - \sum_{t=0}^{\infty}\sum_{j=1}^{J_x} r_t^{*}\hat{k}_{xt}^j - \sum_{t=0}^{\infty}\sum_{j=1}^{J_x} w_t^{*}\hat{n}_{xt}^j$$
$$+ \sum_{t=0}^{\infty}\sum_{i=1}^{I} r_t^{*}\hat{k}_t^i - \sum_{t=0}^{\infty}\sum_{i=1}^{I} p_{xt}^{*}\hat{x}_t^i. \quad (38)$$

First Welfare Theorem: Proof

Now note that:

$$\sum_{t=0}^{\infty} p_{ct}^{*} \sum_{j=1}^{J_c} F_{ct}^{j}(\hat{k}_{ct}^{j}, \hat{n}_{ct}^{j}) - \sum_{t=0}^{\infty} \sum_{j=1}^{J_c} r_t^{*} \hat{k}_{ct}^{j} - \sum_{t=0}^{\infty} \sum_{j=1}^{J_c} w_t^{*} \hat{n}_{ct}^{j} = \sum_{i}^{I} \sum_{t=0}^{\infty} \hat{\pi}_{ct}^{i} \tag{39}$$

$$\sum_{t=0}^{\infty} p_{xt}^{*} \sum_{j=1}^{J_x} F_{xt}^{j}(\hat{k}_{xt}^{j}, \hat{n}_{xt}^{j}) - \sum_{t=0}^{\infty} \sum_{j=1}^{J_x} r_t^{*} \hat{k}_{xt}^{j} - \sum_{t=0}^{\infty} \sum_{j=1}^{J_x} w_t^{*} \hat{n}_{xt}^{j} = \sum_{i}^{I} \sum_{t=0}^{\infty} \hat{\pi}_{xt}^{i} \tag{40}$$

So (38) can be written as:

$$\sum_{i=1}^{I} \sum_{t=0}^{\infty} (p_{ct}^{*} \hat{c}_t^{i} + w_t^{*} \hat{l}_t^{i}) = \sum_{i=1}^{I} \sum_{t=0}^{\infty} (w_t^{*} \bar{n}_t^{i} + r_t^{*} \hat{k}_t^{i} - p_{xt}^{*} \hat{x}_t^{i} + \hat{\pi}_{ct}^{i} + \hat{\pi}_{xt}^{i}) \tag{41}$$

This contradicts (32). Q.E.D.

First Welfare Theorem: Intuition

- In any feasible allocation z, the total cost of the consumption bundle evaluated at prices p^* must be equal to the wealth of households, measured at those prices.
- If z Pareto dominates z^*, then the total cost of z at prices p^*, and therefore households' wealth at those prices, must exceed the total cost of z^* at prices p^* (by preference nonsatiation).
- But by firms' profit maximization, there is no feasible production plan that delivers a value of household wealth at prices p^* higher than that given by z^*.

First Theorem of Welfare Economics

- One of the deepest and most fundamental results in Economics
- Adam Smith (1776) "Invisible Hand", Walras (1870), Edgeworth (1881), etc.
- Follows from a small set of assumptions:
  - Preference nonsatiation
  - market completeness
  - price taking agents
- It says nothing about "desirability" of CE outcomes (e.g. distributional concerns)
- However, very useful to thinking about policy questions:
  - let the market work is always a good policy advice
- Under which circumstances would the 1st Welfare Theorem fail to hold?

Second Welfare Theorem

- The converse is also true: Under additional conditions, a Pareto Optimal allocation can be 'implemented' as a CE.
- This may require a redistribution of initial endowments, or assuming that it is possible to lump-sum transfer across agents. Additionally technical conditions (continuity, differentiability) on U^i and F^j.

Second Welfare Theorem

Theorem 2a. Assume that A1 and A2 hold and that F^j's are continuous, strictly increasing, weakly concave, and all functions are continuously differentiable. Suppose z is a Pareto Optimal allocation, and assume it is interior. Then, there is an alternative assignment of endowments, $\{\hat{k}_0^i, \hat{\bar{n}}_t^i, \hat{\theta}_{ct}^i, \hat{\theta}_{xt}^i\}$ such that:

$$\sum_i \hat{k}_0^i = \sum_i k_0^i,\ \sum_i \hat{\bar{n}}_t^i = \sum_i \bar{n}_t^i,\ \sum_i \hat{\theta}_{ct}^i = \sum_i \hat{\theta}_{xt}^i = 1,$$

and some prices $\{p_{ct}, p_{xt}, r_t, w_t\}$ such that z is a Competitive Equilibrium allocation at the the prices $\{p_{ct}, p_{xt}, r_t, w_t\}$ and initial endowment $\{\hat{k}_0^i, \hat{\bar{n}}_t^i, \hat{\theta}_{ct}^i, \hat{\theta}_{xt}^i\}$.

Second Welfare Theorem

Sketch of Proof:

- Since z is interior, construct prices:

$$p_{c0} = 1,\ p_{ct} = \frac{\partial U^1 / \partial c_t^1}{\partial U^1 / \partial c_0^1},\ w_t = \frac{\partial U^1 / \partial l_t^1}{\partial U^1 / \partial c_0^1}, p_{xt} x_t^1 = r_t k_{xt}^1 + w_t n_{xt}^1$$

  (Note where $p_{xt} x_t^1 = ...$ comes from).

- Since z is Pareto Optimal, it solves (from convexity):

$$\max_{z \in \mathcal{Z}} \sum_i \lambda_i U^i(z^i) \tag{42}$$

  for some $\lambda^i > 0$. Proof: Homework. This implies:

$$\frac{\partial U^i / \partial c_t^i}{\partial U^i / \partial c_0^i} = \frac{\partial U^{i'} / \partial c_t^{i'}}{\partial U^{i'} / \partial c_0^{i'}},\ \text{etc.},\ \forall i, i'$$

Second Welfare Theorem

Sketch of Proof (cont.):

- Given the prices and the allocation, compute value of total spending in the economy W, and the value of individual spending W^i.
- Let $\hat{k}_0^i = \frac{W^i}{W} k_0^i$, $\hat{\bar{n}}_t^i = \frac{W^i}{W} \bar{n}_t^i$, etc.
- With the prices and endowments constructed in this way, need to show that z is a Competitive Equilibrium, i.e. the each agent is maximizing (FOCs, budget constraints).
- Key assumption is convexity: so that we can build prices from marginal conditions and be sure that z is a solution to (42) and marginal conditions are equalized across agents.

Second Welfare Theorem

If lump-sum transfers are available:

Theorem 2b. Assume that A1 and A2 hold, and that F^j's are continuous, strictly increasing, weakly concave, and all functions are continuously differentiable. Suppose z is a Pareto Optimal allocation. Then, there is a choice of lump-sum transfers $\{T^1, ..., T^I\}$, and some prices $\{p_{ct}, p_{xt}, r_t, w_t\}$ such that z is a Competitive Equilibrium with transfers.

Check MWG(1995) Section 16.D. for a similar problem.

Additional reading

References:

- Stokey and Lucas (1989): Chapters 2, 15.
- MWG (1995): Chapter 15, 16.

Alternative Implementations

- In the previous definition of CE, the allocation was "implemented" through a (complete) set of time zero trades.
  - Arrow-Debreu (AD) Equilibrium.
- Recall the budget constraint (8):

$$\sum_{t=0}^{\infty}(p_{ct}c_t^i + p_{xt}x_t^i) \leq \sum_{t=0}^{\infty}(w_t n_t^i + r_t k_t^i + \pi_{xt}^i + \pi_{ct}^i) \qquad (43)$$

- Trades in all goods and all periods settled at time zero, before any consumption and labor take place.
  - Kind of "futures contracts" for all goods and periods.
  - However, there may be implicit borrowing and lending between agents and across time. Why?
- Alternative market structures.

Sequential Market Equilibrium

Simplify the model so that there is no production, no labor;
Instead, one endowment good e_t^i every period; Finite horizon: $t = 0, 1, ..., T$.
Each consumer chooses consumption c_t^i and can borrow or lend an amount L_t^i, at the market interest rate R_t, in order to solve:

$$\max_{\{c_t^i, L_t^i\}_{t=0}^T} U^i(\{c_t^i\}_{t=0}^T) \tag{44}$$

subject to

$$c_i^i + R_{t-1} L_{t-1}^i \leq e_t^i + L_t^i \tag{45}$$

$$L_T^i = 0 \tag{46}$$

Feasibility:

$$\sum_i c_t^i \leq \sum_i e_t^i, t = 0, 1, ... T. \tag{47}$$

Should $L_T^i = 0$ be a restriction, or an equilibrium outcome?

Sequential Market Equilibrium

Definition: A Sequential Market Equilibrium (SME) is a sequence of interest rates $\{R_t\}$, a feasible allocation $\{c_t^i\}$, loan balances $\{L_t^i\}$ for all agents, such that, given the endowment $\{e_t^i\}$ and interest rates, consumption $\{c_t^i\}$ and loan balances $\{L_t^i\}$ solve the individual optimization problems.

SM and AD Equilibrium

Proposition 1. If $\{p_{c0}, ...p_{cT}\}$ and $\{c_0^i, ...c_T^i\}$ are an AD equilibrium, then $\{R_0, ...R_T\}$, $\{L_0^i, ...L_T^i\}$ and $\{c_0^i, ...c_T^i\}$ is a SME with

$$R_t = \frac{p_{ct-1}}{p_{ct}} \tag{48}$$

$$L_\tau^i = \frac{\sum_{t=0}^{\tau} p_{ct}(e_t^i - c_t^i)}{p_{c\tau}} \tag{49}$$

SM and AD Equilibrium

Proposition 2. If $\{R_0, ...R_T\}$, $\{L_0^i, ...L_T^i\}$ and $\{c_0^i, ...c_T^i\}$ is a SM Equilibrium, then $\{p_{c0}, ...p_{cT}\}$ and $\{c_0^i, ...c_T^i\}$ is an AD equilibrium, with

$$p_{c0} = 1, \text{ without loss of generality.} \tag{50}$$

$$p_{ct} = \frac{p_{ct-1}}{R_{t-1}} \tag{51}$$

Aggregation

- Now we turn to additional assumptions that allow us to aggregate the different firms and households.
- Single sector, representative agent, neoclassical growth model.

Aggregation: Common technology and preferences

1. All firms are identical within and across sectors:
   - $F^j_{xt} = F^{j'}_{xt}$, $F^j_{xt} = F^j_{ct}$, $\forall j, j', t$.
2. the common technology F has constant returns to scale:
   - zero profits $\forall j, t$.
3. All households have the same:
   - endowments k^i_0 and $\bar{n}^i_t$
   - preferences U^i
4. U^i strictly concave.
   - unique solution to households problem.

Aggregation: Common technology and preferences

1&2:

$$\sum_j F^j_{ct}(k^j_{ct}, n^j_{ct}) = \sum_j F_c(k^j_{ct}, n^j_{ct}) =$$
$$= J_c \times F_c(k^j_{ct}, n^j_{ct}) = F_c(J_c \cdot k^j_{ct}, J_c \cdot n^j_{ct}) = F_c(k^f_{ct}, n^f_{ct})$$

Similarly for F^j_{xt}, and additionally $F_x = F_c$.

The firm's problem becomes:

$$\max \sum_{t=0}^{\infty} \left(p_{ct} c^f_t + p_{xt} x^f_t - r_t k^f_t - w_t n^f_t \right) \tag{52}$$
$$\text{s.t.} \quad c^f_t + x^f_t \leq F(k^f_t, n^f_t) \ \forall t. \tag{53}$$

$\Rightarrow$ In equilibrium, $p_{ct} = p_{xt}$ if $c^f_t > 0, x^f_t > 0$.

In short:

$$\max \sum_{t=0}^{\infty} \left(p_{ct} F(k^f_t, n^f_t) - r_t k^f_t - w_t n^f_t \right) \tag{54}$$

Aggregation: Common technology and preferences

3 & 4: All HH make the same decisions, every period.

$$\sum_i c_t^i = I \times c_t^1 \tag{55}$$

$$\sum_i x_t^i = I \times x_t^1 \tag{56}$$

From the resource constraint:

$$I \times (c_t^1 + x_t^1) = F(k_t^f, n_t^f) \iff c_t^1 + x_t^1 = F(\frac{k_t^f}{I}, \frac{n_t^f}{I}) \tag{57}$$

Can solve the CE in per capita terms: same as one firm and one household economy.

Aggregation: Common technology and Homothetic preferences

- Aggregation obtains under heterogeneous endowments, but require additional assumption on preferences.
  - Need $U^i = U$ and U is "homothetic":

$$U(a) = U(b) \Rightarrow U(\lambda a) = U(\lambda b),\ \forall \lambda \geq 0. \qquad (58)$$

Proposition: Assume $U^i = U, \forall i$, and that U is homothetic, and F is as above (CRS).
If $\{\{c_t^i, x_t^i, k_t^i, n_t^i, l_t^i\}_{i=1}^I, \{x_t^f, k_t^f, n_t^f\}, \{p_t, r_t, w_t\}\}_{t=0}^\infty$ is a CE of an economy with endowments $\{k_0^i, \bar{n}_t^i\}_{i=1}^I$, then

$$\{\{\sum_i c_t^i, \sum_i x_t^i, \sum_i k_t^i, \sum_i n_t^i, \sum_i l_t^i\}_{i=1}^I, \{x_t^f, k_t^f, n_t^f\}, \{p_t, r_t, w_t\}\}_{t=0}^\infty$$

is a CE in an economy with one consumer with endowments:

$$\bar{k}_0 = \sum_i k_0^i, \bar{n}_t = \sum_i \bar{n}_t^i.$$

Homothetic Preferences: Examples

$$U(a) = U(b) \Rightarrow U(\lambda a) = U(\lambda b), \ \forall \lambda \geq 0.$$

- U is homogeneous of degree α:

$$U(\lambda x) = \lambda^{\alpha} U(x)$$

  - $U(x) = \dfrac{x^{1-\sigma}}{1-\sigma}$

- $U(x, y) = a \log(x) + b \log(y)$

- Non-homothethic:

$$U(x, y) = x + by^{\alpha}$$

A representative household has preferences over streams of a single consumption good c_t given by:

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \tag{59}$$

- $\beta \in (0, 1)$: discount factor.
- U is strictly increasing, twice continuously differentiable, and strictly concave.

Growth Model

Economy: Technology

The technology is:

$$C_t + K_{t+1} = F(K_t, X_t) + (1-\delta)K_t \qquad (60)$$

- Single good produced with F, used for consumption (C_t) or investment (K_{t+1}).
- X_t is a production factor that depends on labor.
- K_t is the **aggregate** stock of physical capital available for production at t,
- $\delta \in (0,1)$ is the depreciation rate of capital,
- $K_{t+1} - (1-\delta)K_t$ is gross investment,
- F is a linearly homogeneous production function with positive and decreasing marginal products: $F_i > 0$, $F_{ii} < 0$, $i = 1, 2$.

Utility maximizing:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{61}$$

subject to

$$c_t + k_{t+1} = r_t k_t + (1-\delta) k_t + \chi_t, \; t = 0, 1, ... \tag{62}$$

where χ_t are income related to labor.

Profit maximizing:

$$\max_{\{K_t, X_t\}} \left(F(K_t, X_t) - r_t K_t - w_t X_t\right) \tag{63}$$

In a competitive equilibrium where the firm rents capital from the household, the rental rate of capitla equals its marginal product:

$$r_t = F_1(K_t, X_t) \tag{64}$$

Since F is homogeneous of degree one:

$$F_1(K, X) = \frac{\partial X f(K/X)}{\partial K} = f'(\hat{K}) \tag{65}$$

where $\hat{K} \equiv \dfrac{K}{X}$ and $f(\hat{K}) = \dfrac{F(K, X)}{X}$. This implies:

$$r_t = f'(\hat{K}_t) \tag{66}$$

Growth Model

Competitive Equilibrium

Household's first order condition with respect to capital is:

$$U'(c_t) = \beta U'(c_{t+1})(r_{t+1} + 1 - \delta). \tag{67}$$

Hence, in a CE, (66) and (67) imply:

$$U'(c_t) = \beta U'(c_{t+1})(f'(\hat{K}_t) + 1 - \delta). \tag{68}$$

With $U(c) = \dfrac{c^{1-\sigma}}{1-\sigma}$ (constant intertemporal elasticity of substitution):

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\Big(f'(\hat{K}_{t+1}) + 1 - \delta\Big) \tag{69}$$

$\Longrightarrow$ Capital accumulation alone (i.e. if $X_t = L$) cannot sustain steady state consumption growth

Growth Model

Exogenous growth

Solow (1956): exogenous labor-augmenting technological change:

$$X_t = A_t L, \text{ with } A_t = (1+\mu)A_{t-1}, \mu \geq 0. \tag{70}$$

L is a fix stock of labor.

Consistent with a steady-state equilibrium where consumption and capital growth at constant rates, with $f'(\hat{K}_t)$ constant.

Guess and verify that:

$$c_{t+1}/c_t = K_{t+1}/K_t = 1+\mu \tag{71}$$

and the optimal ratio $\hat{K}^*$ is given by

$$\Big(1+\mu\Big)^{\sigma} = \beta\Big(f'(\hat{K}^*) + 1 - \delta\Big) \tag{72}$$

Question: is the CE outcome Pareto Optimal?

Growth Model

Spillovers Externality

Assume that the firm face a labor productivity that is proportional to the aggregate ratio of physical capital per worker:

$$X_t = \bar{K}_t L, \text{with } \bar{K}_t = \frac{K_t}{L} \tag{73}$$

Note that the firm takes the productivity as given, i.e. does not take into account the effect of its choices of K_t and X_t on $\bar{K}_t$.

From CE conditions (66) and (67), we obtain now:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\Big(f'(1) + 1 - \delta\Big) \tag{74}$$

Need condition on preferences and techonology for a positive growth rate:

$$\beta\big(f'(1) + 1 - \delta\big) \geq 1. \tag{75}$$

Question: is the CE outcome Pareto Optimal?