In-Class
Lecture

Time Value of Money

Advanced Financial Management

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Key takeaways

01 **Understand the concept of the discount rate or opportunity cost of capital.**

02 **Value a stream of cash flows, either using the future value or the present value.**

Time value of money

- Reward demanded by investors for having an amount of money tied up in an investment

Why is it usually the interest rate?

- Interest rate is your opportunity cost.

- If you do not invest in a project or asset, then you could always deposit it in a bank and earn the market interest rate.

Compounded interest

- *Simple interest* only pays interest on the initial investment.

- ***Compounded interest*** pays interest on the original investment and also on the accumulated interest.

Example

Suppose the semi-annual interest rate r_s is 3%. If I invest €1 today, how much will I have earned at the end of 1 year (assuming compounded interest)?

$$1 + r_s + (1 + r_s) \times r_s = (1 + r_s)^2 = 1.0609$$

What was the return on your investment you *effectively* earned? 6.09%

This is called the Effective Annual Rate = EAR: $(1 + r_s)^2 = 1 + EAR \Leftrightarrow (1 + 3\%)^2 = 1 + EAR \Leftrightarrow r_s = 6.09\%$

Rules of time travel

1. Only cash-flows at the same point in time can be compared or combined
2. To move a cash-flow forward in time, you must compound it:

   $FV_n = C_0 \times (1+r)^n$

3. To move a cash-flow backward in time, you must discount it:

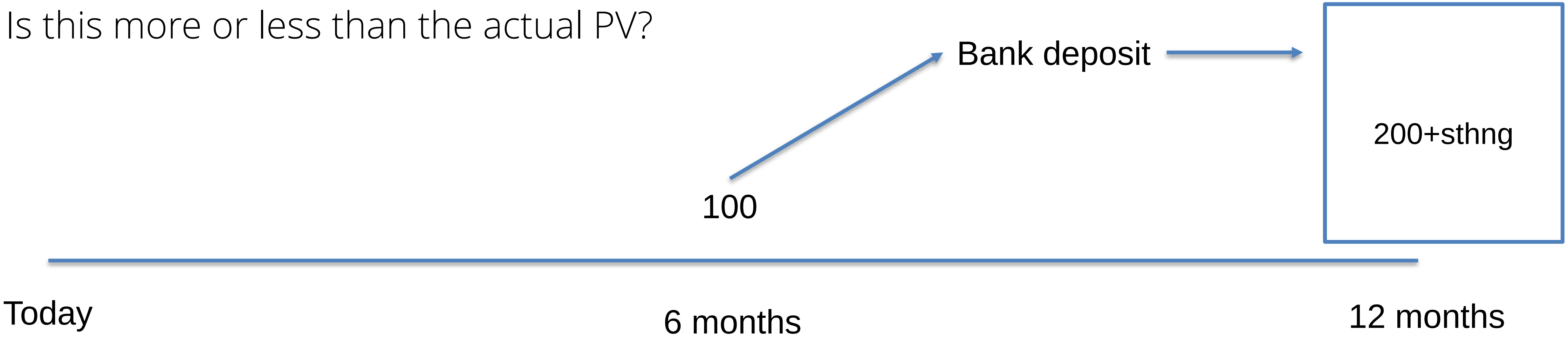
   $PV = C_n / (1+r)^n$

Frequency of payments: example

You receive semi-annual payments of €100 for 1 year. The EAR is 10%.

Is the PV = $\frac{200}{1.1} = 181.81$ correct?

No! Is this more or less than the actual PV?



PV = $\frac{200+sthng}{1.1} > \frac{200}{1.1} = 181.81$

We are going to see three ways of computing the PV

Frequency of payments: example

You receive semi-annual payments of €100 for 1 year. The Effective Annual Rate is 10%.

Computation

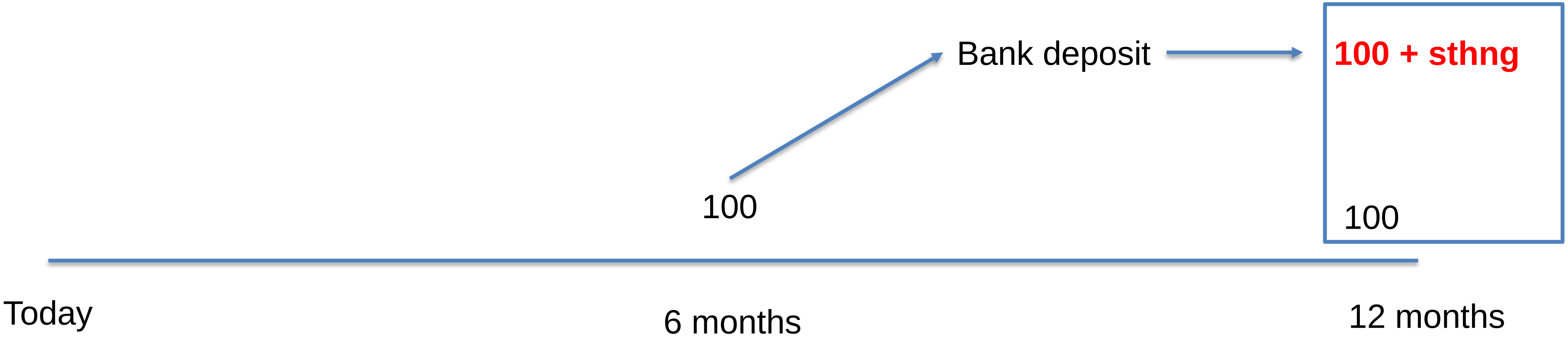
Note that 6 months are ½ of a year:

$$PV = \frac{100}{1.1^{1/2}} + \frac{100}{1.1^{1}} = 186.26$$

CAREFUL: This method only works if we have the Effective Annual Rate

Frequency of payments: example

You receive semi-annual payments of €100 for 1 year. The EAR is 10%.



Frequency of payments: example

You receive semi-annual payments of €100 for 1 year. The Effective Annual Rate is 10%.

Computation

We can compute sthng as the interest payment of a 6-month deposit

$$100 + sthng = 100 \times 1.1^{1/2} = 104.88088 \Rightarrow PV = \frac{100 + 104.88}{1.1} = 186.26$$

CAREFUL: This method only works if we have the Effective Annual Rate

Frequency of payments: example

You receive semi-annual payments of €100 for 1 year. The EAR is 10%.

Computation

1. Compute the effective semi-annual rate:

$$(1 + r_S)^2 = 1 + EAR$$

$$(1 + r_S)^2 = 1 + 10\% \Leftrightarrow 1 + r_S = (1 + 10\%)^{1/2} \Leftrightarrow r_S = 4.9\%$$

2. Compute PV of cash-flows:

$$PV = \frac{100}{1.049} + \frac{100}{1.049^2} = 186.26$$

Present Value and Future Value: other examples

How long until you earn a certain amount?

If we deposit €5000 today in an account paying 10% per year, how many years will it take to grow to €10,000?

$FV_T = C_0(1+r)^T \Leftrightarrow 10{,}000 = 5000(1.1)^T$

$\Leftrightarrow 2 = 1.1^T$

$\Leftrightarrow \ln(2) = T\ln(1.1)$

$\Leftrightarrow T = \frac{\ln(2)}{\ln(1.1)} = 7.27$

$C_0 = Inital\ Investment \quad FV_T = Future\ Value\ at\ T$

$Divide\ both\ sides\ \ by\ 5{,}000$

When we have an unknown in the exponent logs are a useful tool because:

$$\ln(a^x) = x ln(a)$$

$Divide\ both\ sides\ \ by\ \ln(1.1)$

Present Value and Future Value: other examples

What rate do you need to earn a certain amount?

Assume total cost of college education will be €50000 in 12 years. You have €5000 to invest today. What rate of interest must you earn to cover the cost?

$FV_T = C_0(1+r)^T \Leftrightarrow 50{,}000 = 5{,}000\,(1+r)^{12}$

$C_0 = Inital\ Investment \quad FV_{12} = Future\ Value\ at\ T = 12$

$\Leftrightarrow 10 = (1+r)^{12}$

$Divide\ both\ sides\ by\ 5{,}000$

$\Leftrightarrow 10^{1/12} = 1+r$

$We\ raise\ both\ sides\ to\ the\ power\ of\ \frac{1}{12}$

$Remember: \ (x^c)^d = x^{cd} \Rightarrow (x^{12})^{\frac{1}{12}} = x^1 = x$

$\Leftrightarrow r = 21.15\%$

Net present value

- Net present value is the present value of all positive cashflows minus the present value of all negative cashflows
- We never start projects with negative NPV
- If we have to choose a project, we always choose the highest NPV project

Exercise 1

Consider you are offered a project that requires 1000€ today and generates a cashflow of 500€ in one year and 600€ in two years. The rate you can earn in your deposits is 8%.

a. Should you undertake the project?
b. Consider the cashflows for year one and two are the same as before but you are allowed to make the 1,000€ investment at the end of the first year. Should you undertake the project?

Exercise 2

Consider the EAR is 7% and you are offered two projects:

Project A: generates a cashflow of 1000 every year for the next 30 years

Project B: generates a cashflow of 450 every 6 months for the next 30 years

The initial investment is 100 for both projects. Which project would you choose?

Exercise 1

Consider you are offered a project that requires 1000€ today and generates a cashflow of 500€ in one year and 600€ in two years. The rate you can earn in your deposits is 8%.

a. Should you undertake the project?

$$NPV = -1000 + \frac{500}{1.08} + \frac{600}{1.08^2} \approx -23$$

We do not undertake the project because the NPV is negative

b. Consider the cashflows for year one and two are the same as before but you are allowed to make the 1,000€ investment at the end of the first year. Should you undertake the project?

Year 0 Year 1 Year 2

$$NPV = 0 + \frac{500 - 1000}{1.08} + \frac{600}{1.08^2} = -\frac{500}{1.08} + \frac{600}{1.08^2} \approx 51$$

We undertake the project because the NPV is positive

Exercise 2

Consider the EAR is 7% and you are offered two projects:

Project A: generates a cashflow of 1000 every year for the next 30 years

Project B: generates a cashflow of 450 every 6 months for the next 30 years

The initial investment is 100 for both projects. Which project would you choose?

We know the initial cashflow is the same and they follow a similar patter. The questions is whether 450 in 6 months and 450 in a year is better than 1000 in a year (then it repeats 29 times):

$$PV_A(1year) = -100 + \frac{1000}{1.07} = 834.6$$

$$PV_B(1year) = -100 + \frac{450}{1.07^{\frac{1}{2}}} + \frac{450}{1.07} = 755.6$$

$$We\ choose\ project\ A$$

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Exercise 2 (Math solution)

Consider the EAR is 7% and you are offered two projects:

Project A: generates a cashflow of 1000 every year for the next 30 years

Project B: generates a cashflow of 450 every 6 months for the next 30 years

The initial investment is 100 for both projects. Which project would you choose?

We know we choose project A only if $NPV_A > NPV_B \Rightarrow NPV_A - NPV_B > 0$

$$NPV_A = -100 + 0 + \frac{1000}{1.07} + 0 + \frac{1000}{1.07^2} + \cdots + \frac{1000}{1.07^{30}}$$

$$NPV_B = -100 + \frac{450}{1.07^{\frac{1}{2}}} + \frac{450}{1.07} + \frac{450}{1.07^{\frac{3}{2}}} + \frac{450}{1.07^2} + \cdots + \frac{450}{1.07^{30}}$$

$$NPV_A - NPV_B = -\frac{450}{1.07^{\frac{1}{2}}} + \frac{550}{1.07} - \frac{450}{1.07^{\frac{3}{2}}} + \frac{550}{1.07^2} + \cdots + \frac{550}{1.07^{30}}$$

Exercise 2 (Math solution)

$$NPV_A - NPV_B = -\frac{450}{1.07^{\frac{1}{2}}} + \frac{550}{1.07} - \frac{450}{1.07^{\frac{3}{2}}} + \frac{550}{1.07^2} + \cdots + \frac{550}{1.07^{30}}$$

$$NPV_A - NPV_B = \left(-\frac{450}{1.07^{\frac{1}{2}}} + \frac{550}{1.07}\right)\underbrace{\left(1 + \frac{1}{1.07} + \frac{1}{1.07^2} + \cdots + \frac{1}{1.07^{29}}\right)}_{\text{Positive}}$$

$$NPV_A - NPV_B > 0 \Leftrightarrow -\frac{450}{1.07^{\frac{1}{2}}} + \frac{550}{1.07} > 0$$

$$\Leftrightarrow \frac{550}{1.07} > \frac{450}{1.07^{\frac{1}{2}}}$$

$$\Leftrightarrow \frac{550}{450} > 1.07^{1/2}$$

$$\textit{Since the inequality holds } (1.22 > 1.03), \textit{we choose project } A$$