Microeconomic Theory II

Spring 2024

Final Exam

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You have a total of 120 minutes (2 hours) to solve the exam.
Identify each sheet with your Student Number and Name.
Good luck!

I (4.5 points)

Consider a market with a monopolist that produces a good of variable quality $q \geq 0$ at cost $C(q)$ per unit of good (priced at p). Each customer wants to buy one unit of the good and is characterized by her taste for quality θ, where $\theta \in \{\theta_L, \theta_H\}$ with $\theta_L < \theta_H$. An agent's utility will therefore be $\theta.q - p$. Each customer knows her θ, but the monopolist only knows that the proportion of agents with $\theta = \theta_H$ is equal to λ.

a. Monopolist solves $max_{q,p} p - C(q)$ s.t. $\theta.q - p \geq 0$, yielding $\theta = C'(q)$ and $\theta.q = p$
b. Monopolist now solves max $\lambda (p_H - C(q_H)) + (1-\lambda)(p_L - C(q_L))$
s.t.
(IC_H) $\theta_H q_H - p_H \geq \theta_H q_L - p_L$
(IC_L) $\theta_L q_L - p_L \geq \theta_L q_H - p_H$
(IR_H) $\theta_H q_H - p_H \geq 0$
(IR_L) $\theta_L q_L - p_L \geq 0$

We can show that IC_H and IR_L will bind (and imply IR_H)
Therefore at the optimum $p_L = \theta_L q_L$ and $p_H = \theta_H q_H + (\theta_L - \theta_H) q_L$
Replacing in the objective function we have:
max $\lambda (\theta_H q_H + (\theta_L - \theta_H) q_L - C(q_H)) + (1-\lambda)(\theta_L q_L - C(q_L))$ yielding $\theta_H = C'(q_H)$ *(efficiency at the top)*
and $\lambda (\theta_H - \theta_L) = (1-\lambda)(\theta_L - C'(q_L))$ and $C'(q_L) = \theta_L - \frac{\lambda}{1-\lambda}(\theta_H - \theta_L)$ implicitly defines q_L

Since $p_H = \theta_H q_H + (\theta_L - \theta_H) q_L$, there is an informational rent for type H who receives $\theta_H q_H - p_H = (\theta_H - \theta_L) q_L$

c. Serving only type H would lead to $\theta_H q_H = p_H$ and $\theta_H = C'(q_H)$. Expected profits would become $\lambda (\theta_H q_H - C(q_H))$ instead of $\lambda (\theta_H q_H + (\theta_L - \theta_H) q_L - C(q_H)) + (1-\lambda)(\theta_L q_L - C(q_L))$. Depending on the parameters and C, it might be better to serve only one type.

II (5.5 points)

(a) We don't need to worry about IR/BB. We can use the Groves-Clark mechanism to achieve dominant-strategy implementation and efficiency:

- the decision function $x^*(\widehat{\theta})$ is implicitly defined by the FOC of the social surplus maximization problem

$$\max_x \sum_{i=1}^{I} v_i(x, \theta_i) + v_0(x, \theta) = \max_x \sum_{i=1}^{I} \theta_i v(x) - cx \Rightarrow v'(x^*) = \frac{c}{\sum_{i=1}^{I} \theta_i}$$

- each agent's transfer $t_i(\widehat{\theta}) = \sum_{j \neq i} \widehat{\theta}_j v(x^*(\widehat{\theta}_i, \widehat{\theta}_{-i})) - cx^*(\widehat{\theta}_i, \widehat{\theta}_{-i}) + \tau_i(\widehat{\theta}_{-i})$

and the proof that it is a dominant strategy to report truthfully is straightforward.

(b) With IR but no BB, the Government can simply make arbitrarily large transfers (using the $\tau_i(\widehat{\theta}_{-i})$ component of the transfer) to ensure participation.

(c) With BB, we need to use the AGV mechanism. The decision function is the same but each agent's transfer is now:

$$t_i(\widehat{\theta}_i) = E_{\theta_{-i}}[\sum_{j\neq i} \theta_j v(x^*(\widehat{\theta}_i, \theta_{-i})) - cx^*(\widehat{\theta}_i, \theta_{-i})] + \tau_i(\widehat{\theta}_{-i})$$

Once again, the proof that we have Bayesian IC is straightforward. BB is also met if we set

$\tau_i(\widehat{\theta}_{-i}) = -\sum_{j\neq i} \frac{E_{\theta_{-j}}(\sum_{k\neq j} \theta_k v(x^*(\widehat{\theta}_j, \theta_{-j})))}{I-1} + \sum_{j\neq i} \frac{E_{\theta_{-j}}(cx^*(\widehat{\theta}_j, \theta_{-j}))}{I}$

so that

$$\sum_i E_{\theta_{-i}}[\sum_{j\neq i} \theta_j v(x^*(\widehat{\theta}_i, \theta_{-i})) - cx^*(\widehat{\theta}_i, \theta_{-i})] - \sum_i \sum_{j\neq i} \frac{E_{\theta_{-j}}\left(\sum_{k\neq j} \theta_k v(x^*(\widehat{\theta}_j, \theta_{-j}))\right)}{I-1} + \sum_i \sum_{j\neq i} \frac{E_{\theta_{-j}}(cx^*(\widehat{\theta}_j, \theta_{-j}))}{I}$$

$$= -\sum_i E_{\theta_{-i}}(cx^*(\widehat{\theta}_i, \theta_{-i})) + \sum_i \sum_{j\neq i} \frac{E_{\theta_{-j}}(cx^*(\widehat{\theta}_j, \theta_{-j}))}{I} = -\frac{1}{I} \sum_i E_{\theta_{-i}}(cx^*(\widehat{\theta}_i, \theta_{-i}))$$

III (6 points)

a) Let θ and ϕ respectively denote the buyer's and the seller's valuations, where both θ and ϕ follow a $U[0,2]$ distribution. Let Y_B denote the transfer from the buyer, Y_S denote the transfer to the seller and Q denote the decision function (probability of trade).

Let $q_S(\phi) = E_\theta(Q(\theta, \phi))$ and $y_S(\phi) = E_\theta(Y_S(\theta, \phi))$ and let $q_B(\theta) = E_\phi(Q(\theta, \phi))$ and $y_B(\theta) = E_\phi(Y_B(\theta, \phi))$.

The profits of a seller are $\pi_S = y_S - \phi q_S$ and the profits of the buyer are $\pi_B = \theta x_B - y_B$.

(i) IC

For the seller's IC, from $\pi_S(\widehat{\phi}, \phi) = y_S(\widehat{\phi}) - \phi q_S(\widehat{\phi})$ we can write the value function $\pi_S(\phi) \equiv \pi_S(\phi, \phi)$. Using the value function and applying the envelope theorem (or just incorporating the FOC), we have that $\frac{d\pi_S}{d\phi}(\phi) = \frac{\partial \pi_S}{\partial \phi}(\phi)$ and

$$\frac{d\pi_S}{d\phi}(\phi) = -q_S(\phi).$$

Since $\frac{d}{d\phi}\left(\frac{\frac{\partial \pi_S}{\partial q_S}}{\frac{\partial \pi_S}{\partial y_S}}\right) = -1 < 0$, CS^- holds. Therefore, we need $q_S(\phi)$ to be nonincreasing in any implementable contract.

$$IC \iff \begin{cases} \frac{d\pi_S}{d\phi}(\phi) = -q_S(\phi) \\ q_S(\phi) \text{ nonincreasing} \end{cases}$$

We can now rewrite the profits of the seller of type ϕ as the sum of the profits of the "worst type" and an integral. Integrating $\frac{d\pi_S}{d\phi}(\phi) = -q_S(\phi)$ from ϕ to $\overline{\phi} = 2$ yields $\pi_S(\phi) = \pi_S(2) + \int_\phi^2 q_S(\widetilde{\phi})d\widetilde{\phi}$.

Since $\frac{d}{d\theta}\left(\frac{\frac{\partial \pi_B}{\partial q_B}}{\frac{\partial \pi_B}{\partial(-y_B)}}\right) = 1 > 0$, CS^+ holds. Therefore, we need $q_B(\theta)$ to be nondecreasing in any implementable contract.

$$IC \iff \begin{cases} \frac{d\pi_B}{d\theta}(\theta) = q_B(\theta) \\ q_B(\theta) \text{ nondecreasing} \end{cases}$$

We can now rewrite the profits of the buyer of type θ as the sum of the profits of the "worst type" and an integral. Integrating $\frac{d\pi_B}{d\theta}(\theta) = q_B(\theta)$ from $\underline{\theta} = 0$ to θ yields $\pi_B(\theta) = \pi_B(0) + \int_0^\theta q_B(\widetilde{\theta})d\widetilde{\theta}$.

(ii) IR just requires that $\pi_B(0) \geq 0$ and $\pi_S(2) \geq 0$.

(iii) Ex post efficiency implies that:

$$Q(\theta, \phi) = \begin{cases} 1 & \text{if } \theta \geq \phi \\ 0 & \text{if } \theta < \phi \end{cases}$$

yielding $q_S(\phi) = 1 - \frac{\phi}{2}$ and $q_B(\theta) = \frac{\theta}{2}$. Combining (i), (ii) and (iii) yields: $\pi_B(\theta) \geq \int_0^\theta \frac{\tilde{\theta}}{2} d\tilde{\theta}$ and $\pi_S(\phi) \geq \int_\phi^2 \left(1 - \frac{\tilde{\phi}}{2}\right) d\tilde{\phi}$

But then $E_\theta[\pi_B(\theta)] + E_\phi[\pi_s(\phi)] \geq \frac{1}{3} + \left(1 - 1 + \frac{1}{3}\right) = \frac{2}{3}$

b) Under truth-telling and with $q_S(\phi) = 1 - \frac{\phi}{2}$ and $q_B(\theta) = \frac{\theta}{2}$, we have $\pi_S(\phi) = y_s(\phi) - \phi\left(1 - \frac{\phi}{2}\right)$ and $\pi_B(\theta) = \frac{\theta^2}{2} - y_B(\theta)$.

But then $E_\theta[\pi_B(\theta)] + E_\phi[\pi_s(\phi)] = E_\phi[y_s(\phi)] - E_\theta[y_B(\theta)] - E_\phi\left[\phi\left(1 - \frac{\phi}{2}\right)\right] + E_\theta\left[\frac{\theta^2}{2}\right] =$

$$= E_\phi[y_s(\phi)] - E_\theta[y_B(\theta)] - \frac{1}{3} + \frac{2}{3} = E_\phi[y_s(\phi)] - E_\theta[y_B(\theta)] + \frac{1}{3}$$

For budget balance, $Y_B \geq Y_s$ and $E_\phi[y_s(\phi)] - E_\theta[y_B(\theta)] \leq 0$. But then $E_\theta[\pi_B(\theta)] + E_\phi[\pi_s(\phi)] \leq \frac{1}{3}$.

IV (4 points)

(a) Pareto optimality is satisfied: if aP_ib for all b and for all i then $u_i(a) > u_i(b)$ for all b and for all i and $\sum u_i(a) > \sum u_i(b)$ for all y which is equivalent to having $aF_P(R)b$ for all b.

(b) Independence of irrelevant utilities is also satisfied: the social ordering of a and b depends only on $\sum u_i(a)$ and $\sum u_i(b)$.

(c) There is no dictatorship; in fact, we have anonymity (we can have a permutation of all agents and the sum of the utilities for each alternative would remain the same).

(d) The answers to (a), (b) and (c) are compatible with Arrow's Theorem because the theorem applies to social welfare functionals that associate a preference profile $R \in \Re^I$ with a social preference relation in $\Re$. Here, we have a function that associates a profile of utilities with a social preference relation; this means that the same individual preferences can give rise to several different utility values associated with the alternatives and therefore the same preference profile can result in several different social preference relations.