Microeconomic Theory II

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Final Exam

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You have a total of 120 minutes (2 hours) to solve the exam.
Identify each sheet with your Student Number and Name.
Good luck!

I (5 points)

1.

(a) Under symmetric information, the pay-off is constant and is determined by the participation constraint: $w^L = \underline{w}$ and $w^H = (\sqrt{\underline{w}} + v)^2$. The owner demands the high effort if $p(\pi_2 - \pi_1) \geq (\sqrt{\underline{w}} + v)^2 - \underline{w}$.

(b) With the same wage $\underline{w}$ the manager would spontaneously exert low effort. The optimal contract that achieves high effort is $w_1 = \underline{w}$ and $w_2 = (\frac{v}{p} + \sqrt{\underline{w}})^2$.

(c) The owner will demand effort e^H if $p\pi_2 + (1. - p)\pi_1 - p(\frac{v}{p} + \sqrt{\underline{w}})^2 - (1-p)\underline{w} \geq \pi_1 - \underline{w}$ i.e. if $\pi_2 - \pi_1 \geq (\frac{v}{p} + \sqrt{\underline{w}})^2 - \underline{w}$.

II (6.5 points)

a.

Let t denote transfers and x denote the decision function. The utility of a supplier is $u_i = t_i - x_i\theta_i$. With $X_i(\theta_i) = \int_{\theta_{-i}} x_i(\theta)dF(\theta_{-i})$ and $T_i(\theta_i) = \int_{\theta_{-i}} t_i(\theta)dF(\theta_{-i})$, we have $U_i(\widehat{\theta}_i, \theta_i) = T_i(\widehat{\theta}_i) - X_i(\widehat{\theta}_i)\theta_i$ and the FOC of the problem $\max_{\widehat{\theta}_i} U_i(\widehat{\theta}_i, \theta_i)$ must be met for $\widehat{\theta}_i = \theta_i$ for the "local IC" to hold. We then have

$$\frac{\partial T_i}{\partial \widehat{\theta}_i}(\theta_i) - \theta_i \frac{\partial X_i}{\partial \widehat{\theta}_i}(\theta_i) = 0.$$

Let $U_i(\theta_i) \equiv U_i(\theta_i, \theta_i)$. Using the value function and incorporating the FOC (or just applying the envelope theorem), we have that $\frac{dU_i}{d\theta_i}(\theta_i) = \frac{\partial U_i}{\partial \theta_i}(\theta_i)$ and

$$\frac{dU_i}{d\theta_i}(\theta_i) = -X_i(\theta_i).$$

Since $\frac{d}{d\theta_i}\left(\frac{\frac{\partial U_i}{\partial X_i}}{\frac{\partial U_i}{\partial T_i}}\right) = -1 < 0, CS^-$ holds. Therefore, we need $X_i(\theta_i)$ to be nonincreasing in any implementable contract.

$$IC \iff \begin{cases} \frac{dU_i}{d\theta_i}(\theta_i) = -X_i(\theta_i) \\ X_i(\theta_i) \text{ nonincreasing} \end{cases}$$

b.

Integrating $\frac{dU_i}{d\theta_i}(\theta_i) = -X_i(\theta_i)$ from θ_i to $\overline{\theta}_i$ yields:

$U_i(\theta_i) = U_i(\overline{\theta}_i) + \int_{\theta_i}^{\overline{\theta}_i} X_i(\widetilde{\theta}_i)d\widetilde{\theta}_i$

Combining $U_i(\theta_i) = T_i(\theta_i) - X_i(\theta_i)\theta_i$ and $U_i(\theta_i) = U_i(\overline{\theta}_i) + \int_{\theta_i}^{\overline{\theta}_i} X_i(\widetilde{\theta}_i)d\widetilde{\theta}_i$ yields $T_i(\theta_i) = X_i(\theta_i)\theta_i + U_i(\overline{\theta}_i) + \int_{\theta_i}^{\overline{\theta}_i} X_i(\widetilde{\theta}_i)d\widetilde{\theta}_i$. Since $T_i(.)$ enters the principal's objective function with a negative sign, the principal will want to set $U_i(\overline{\theta}_i) = 0$ (IR is binding for the "worst type").

The principal's problem is

$$\max_{\{x_i(\theta)\}_{i=1,2}} \sum_i E_{\theta_i}\left[v.X_i(\theta_i) - X_i(\theta_i)\theta_i - \int_{\theta_i}^{\overline{\theta}_i} X_i(\widetilde{\theta}_i)d\widetilde{\theta}_i\right]$$

(subject to the monotonicity constraints)

Since

$$E_{\theta_i}\left[\int_{\theta_i}^{\overline{\theta}_i} X_i(\widetilde{\theta}_i)d\widetilde{\theta}_i\right] = \int_{\underline{\theta}_i}^{\overline{\theta}_i}\int_{\theta_i}^{\overline{\theta}_i} X_i(\widetilde{\theta}_i)d\widetilde{\theta}_i f(\theta_i)d\theta_i,$$

integration by parts yields

$$E_{\theta_i}\left[\int_{\theta_i}^{\overline{\theta}_i} X_i(\widetilde{\theta}_i)d\widetilde{\theta}_i\right] = E_{\theta_i}\left[X_i(\theta_i).\frac{F(\theta_i)}{f(\theta_i)}\right]$$

The simplified problem is then:

$$\max_{\{x_i(\theta)\}_{i=1,2}} \sum_i E_{\theta_i}\left[X_i(\theta_i)(v - \theta_i - \frac{F(\theta_i)}{f(\theta_i)}\right] = \max_{\{x_i(\theta)\}_{i=1,2}} \sum_i E_{\theta}\left[x_i(\theta)(v - \theta_i - \frac{F(\theta_i)}{f(\theta_i)}\right]$$

Let $J(\theta_i) \equiv v - \theta_i - \frac{F(\theta_i)}{f(\theta_i)}$. Given the assumption, $J(.)$ is decreasing. The optimal mechanism is the one that sets $x_i(\theta) = 1$ if $J(\theta_i) \geq 0$ and $J(\theta_i) \geq J(\theta_j)$ (and 0 otherwise). Letting ε be such that $J(\varepsilon) = 0$, the optimal mechanism is: $x_i(\theta) = \begin{cases} 1 & \text{if } \theta_i = \min\{\theta_i, \theta_j, \varepsilon\} \\ 0 & \text{otherwise} \end{cases}$

c.

We could have a second-price "auction" where the lowest bid wins and receives the highest bid (or ε, whichever is lowest).

III (4.5 points)

AGV mechanism:

$x^*(\hat{\theta})$ maximizes $\sum_{i=0}^{I} v_i(x, \hat{\theta}_i)$

$t_i(\hat{\theta})= E_{\theta_{-i}}[\sum_{j\neq i} v_j(x^*(\hat{\theta}_i, \theta_{-i}), \theta_j) \,|\theta_i] + \tau_i(\hat{\theta}_{-i})$

$\hat{\theta}_i = \theta_i$ maximizes $E_{\theta_{-i}}[v_i(x^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + \sum_{j\neq i} v_j(x^*(\hat{\theta}_i, \theta_{-i}), \theta_j) \,|\theta_i]$

In this case,

$$x^*(\hat{\theta}) = \begin{cases} 1 \; if \; \sum_{i=1}^{I} \hat{\theta}_i \geq c \\ 0 \; if \; \sum_{i=1}^{I} \hat{\theta}_i < c \end{cases}$$

And

$$t_i(\hat{\theta}) = \begin{cases} E_{\theta_{-i}}\left[\sum_{j\neq i} \theta_j \,|\theta_i\right] - c + \tau_i(\hat{\theta}_{-i}) = I - 1 - c + \tau_i(\hat{\theta}_{-i}) \; if \; \hat{\theta}_i + I - 1 \geq c \\ \tau_i(\hat{\theta}_{-i}) \; if \; \hat{\theta}_i + I - 1 < c \end{cases}$$

Bayesian IC is satisfied:

$\hat{\theta}_i = \theta_i$ maximizes $\theta_i E_{\theta_{-i}}[(x^*(\hat{\theta}_i, \theta_{-i}))] + I - 1 + \tau_i(\hat{\theta}_{-i})$

Budget balance is ensured in expectation:

Let $\varepsilon_i(\hat{\theta}_i) \equiv \begin{cases} I - 1 - c \; if \; \sum_{i=1}^{I} \hat{\theta}_i \geq c \\ 0 \; if \; \sum_{i=1}^{I} \hat{\theta}_i < c \end{cases}$

If $I < c$, provision is not expected (no cost and no transfers) and therefore $\tau_i(\hat{\theta}_{-i}) = 0$ for all i.

If $I \geq c$, provision is expected (generating a total cost of $I.(I-1-c) + c = (I-c)(I-1)\;c$). In that case, we therefore need to set $\tau_i(\hat{\theta}_{-i}) = (I-c)(I-1)/I$ for all i.

IV (4 points)

(a) According to the majority rule, the social preference will be $SE \succ AE \succ NE$ ($SE \succ AE$; $AE \succ NE$; and $SE \succ NE$).

According to the Borda rule, we have $NE \succ SE \succ AE$ (with scores, respectively, of 147, 140, and 113).
Group A prefers dictatorship, Group B prefers Borda, Group C prefers majority.

(b)

The social preference that results from majority is rational (complete, transitive and reflexive). This does not contradict Arrow's Theorem because the rationality of the social preference will not hold for all possible preference profiles (we go back to the Condorcet paradox).

(c)

Strategy-proofness means that no agent can have an incentive to misrepresent her preferences, regardless of what other agents are announcing. Therefore, it is enough to find an example of announcements such that one agent prefers to announce a preference relation different from her own. Let all agents in groups B and C announce truthfully. If 27 agents in group A are also announcing truthfully but 3 agents in group A are announcing the preferences of group C, then the 31st agent in group A would rather announce the preferences of group C as well (and lead to the choice of SE) instead of announcing her true preferences (leading to the choice of NE).