> You have exactly 2 hours to finish the exam. Please try to complete the exam as much as possible with the time given to you. Make sure you show all the crucial steps that led to the final answer of each question in your answer sheet. Interpretation is part of the examination. Good luck!

1. (25 pts) Consider the following ARMA(1,2) time series model where ε_t follows a white noise process:

$$y_t = 2 - 0.6y_{t-1} + \varepsilon_t + 0.8\varepsilon_{t-1} + 0.07\varepsilon_{t-2} \qquad (1)$$

(a) (10 pts) Find the homogeneous solution and the particular solution. You can use the lag operator without solving for the coefficients explicitly. Show what is the final solution for (1).

The homogeneous equation is given by $y_t^h = -0.6y_{t-1}^h$. *The homogeneous solution is given by* $y_t^h = -A0.6^t$.

The particular solution using the lag operator is given by $y_t^p = \frac{2+(1+0.8L+0.07L^2)\varepsilon_t}{1+0.6L}$.

The general or final solution is given by the sum of the homogeneous solutions plus the particular solutions: $y_t = -A0.6^t + \frac{2+(1+0.8L+0.07L)\varepsilon_t}{1+0.6L}$.

(b) (5 pts) Is (1) stationary and/or invertible?

The ARMA model is stationary since the coefficient on the autoregressive component is less than 1. In this case it is negative, -0.6, which means that the solution converges in an oscillatory fashion.

In order to check invertibility we need to solve for the lag polynomial associated with the moving average component, $(1 + 0.8L + 0.07L^2)$. *We find that the roots of the inverted characteristic equation are:* $z_1 = \frac{-0.8+0.4}{0.14} \approx -2.86$ *and* $z_2 = \frac{-0.8-0.4}{0.14} \approx -8.57$. *Since the absolute value of both roots is strictly larger than 1, we have that the process is invertible.*

(c) (5 pts) Find $E[y_t]$ and $var[y_t]$.

Using b) we know that the process is stationary. This implies that the first and second moments are constant for all t. We can use this to find the unconditional mean of y_t*:*

$$E[y_t] = 2 - 0.6E[y_t] \Rightarrow E[y_t] = \frac{2}{1+0.6} = 1.25 \qquad (1)$$

As for the variance: $var(y_t) = -0.6^2 var(y_t) + \sigma^2 + 0.8^2\sigma^2 + 0.07^2\sigma^2 + 2(-0.6)0.8cov(y_{t-1}, \varepsilon_{t-1}) + 2(-0.6)0.07cov(y_{t-1}, \varepsilon_{t-2})$.

Hence we have that:

$$var(y_t) = \frac{\sigma^2(1 + 0.64 + 0.0049 - 2(-0.6)0.8 - 2(0.6)0.07(0.8))}{0.64} \approx 0.965\sigma^2 \qquad (2)$$

(d) (5 pts) Compute the ACF for the first and second lag, ρ_1 and ρ_2, respectively.

Let's put the process in terms of the unconditional mean and then multiply both sides by $y_{t-1} - \mu$ *and take expectations. We get that* $\gamma_1 = -0.6\gamma_0 + 0.8\sigma^2 + 0.07 \times 0.8\sigma^2$.

Hence, $\rho_1 = -0.6 + 0.856/0.965 \approx 0.287$

WE can proceed in the same way and multiply both sides of the process with the unconditional mean representation by $y_{t-2} - \mu$ *and take expectations.*

We get that $\gamma_2 = -0.6\gamma_1 + 0.07\sigma^2$. *Hence,* $\rho_2 = -0.6 \times 0.287 + 0.07/0.965 \approx -0.01$

2. (20 pts) True or False. Justify your answer.

   (a) (5 pts) For an AR model, we can use the PACF to identify the number of lags, and for a MA model we can use the ACF to identify the number of lags.

   True, the ACF is not informative about the number of lags in a AR model since the ACF can decay slowly with any number of lags. The PACF on the other hand measures how much information an additional lag adds in explaining y_t and can be used to identify the number of lags. Specifically, we pick the number of lags by finding the last lag that is significant in the PACF. The PACF, on the other hand, is not useful to select the number of lags in the MA models since there the PACF decays slowly for any number of lags in the moving average component. However, the ACF is directly affected by the number of lags, and so we can use it to perfectly identify how many and what lags to include.

   (b) (5 pts) Suppose we have estimated two models, and that f_1 and f_2 are the forecasts of each model. The variance of the combined forecast error is always smaller than the variance of each individual forecast error.

   False. The variance of the combined forecast error can be greater than each individual forecast. This depends on two forces: the weights of each model and the covariance between the forecast errors of the two models.

   (c) (5 pts) We need to identify a VAR before using it to make forecasts.

   False. $X_t = A_1 X_{t-1} + e_t$. *Hence,* $E[X_{t+1}] = A_1 X_t$. *Hence, there is no need to identify the VAR model before making forecasts. The VAR in the reduced form is enough to make forecasts.*

   (d) (5 pts) Under the Cholesky identification approach to a VAR, the variance of each structural shock is always 1.

   True. The Cholesky always decomposes the variance-covariance matrix $\Sigma = LL'$. *Using* $e_t = L\varepsilon_t$, *and so* $\varepsilon_t = L^{-1}e_t$.
   The variance-covariance matrix of the structural shocks is then $\Sigma_\varepsilon = L^{-1}\Sigma L^{-1'} = L^{-1}LL'L^{-1'} = I$

3. (40 pts) (VAR) Suppose we have the following VAR with two variables, Stock Prices S_t and Interest Rates I_t:

$$I_t = 2 + 0.1I_{t-1} + 0.4S_{t-1} + e_{1t}$$
$$S_t = 1 + 0.9I_{t-1} + 0.3S_{t-1} + e_{2t}$$

and that the residual variance-covariance matrix is given by:

$$\Sigma = \begin{bmatrix} 0.1 & -0.3 \\ -0.3 & 1.1 \end{bmatrix}$$

   (a) (10 pts) Find the two periods ahead conditional forecast of stock prices, $E_t[S_{t+2}]$.

   The two step-ahead forecast for stock prices is given by $E_t[S_{t+2}] = 1 + 0.9E_t[I_{t+1}] + 0.3E_t[S_{t+1}]$.
   Hence, first we need to find the one step-ahead forecasts for both stock prices and interest rates.
   $E_t[S_{t+1}] = 1 + 0.9I_t + 0.3S_t$. *And* $E_t[I_{t+1}] = 2 + 0.1I_t + 0.4S_t$.
   Lets substitute that into the two step forecast to find:

$$\begin{aligned} E_t[S_{t+2}] &= 1 + 0.9(1 + 0.9I_t + 0.3S_t) + 0.3(2 + 0.1I_t + 0.4S_t) \\ E_t[S_{t+2}] &= 3.1 + 0.36I_t + 0.45S_t \end{aligned}$$

Cont.

(b) (10 pts) Write the structural VAR representation. Assume interest rates are not affected contemporaneously by stock prices . Show your matrix B after imposing this restriction.

Let $X_t = \begin{bmatrix} I_t \\ S_t \end{bmatrix}$. *The structural VAR representation is given by*

$$BX_t = \Gamma_0 + \Gamma_1 X_{t-1} + \varepsilon_t$$

The matrix B specifies how each variable affect each other contemporaneously.

$$\begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix} \begin{bmatrix} I_t \\ S_t \end{bmatrix}$$

Assuming that interest rates are not affected by stock prices is the same as assuming that $b_{12} = 0$. *Hence, B becomes:*

$$\begin{bmatrix} 1 & 0 \\ b_{21} & 1 \end{bmatrix}$$

(c) (10 pts) Under the restriction imposed in c), identify B and the variance of the structural shocks.

Mapping the structural VAR representation into the reduced form we know that $e_t = B^{-1}\varepsilon_t$. $B^{-1} = \begin{bmatrix} 1 & 0 \\ -b_{21} & 1 \end{bmatrix}$. *Hence, we have*

$$\begin{aligned} e_{1t} &= \varepsilon_{It} \\ e_{2t} &= -b_{21}\varepsilon_{It} + \varepsilon_{St} \end{aligned}$$

Taking the variance on this system we get:

$$\begin{aligned} var(e_{1t}) &= \sigma^2_{It} \\ var(e_{2t}) &= b^2_{21}\sigma^2_{It} + \sigma^2_{St} \\ cov(e_{1t}, e_{2t}) &= -b_{21}\sigma^2_{It} \end{aligned}$$

The variance and covariance of the reduced form errors is given and we can substitute them into the system to get

$$\begin{aligned} 0.1 &= \sigma^2_{It} \\ 1.1 &= b^2_{21}\sigma^2_{It} + \sigma^2_{St} \\ -0.3 &= -b_{21}\sigma^2_{It} \end{aligned}$$

So, we have a system with 3 equations and 3 unknowns. We find that $\sigma^2_{It} = 0.1$, $\sigma^2_{St} = 0.2$ *and* $b_{21} = 3$

(d) (10 pts) Find the contemporaneous and one step ahead impulse responses using what you found in c). Interpret the results.

The impulse response functions are easy to compute once we use the VMA(∞) representation of the VAR:

$$X_t = \mu + e_t + A_1 e_{t-1} + A_1 e_{t-2} + ...$$

Using the structural shocks:

$$X_t = \mu + B^{-1}\varepsilon_t + A_1 B^{-1}\varepsilon_{t-1} + A_1 B^{-1}\varepsilon_{t-2} + ...$$

Hence, $\frac{\partial X_t}{\partial \varepsilon_t} = B^{-1}$ *and* $\frac{\partial X_{t+1}}{\partial \varepsilon_t} = A_1 B^{-1}$

$$A_1 B^{-1} = \begin{bmatrix} 0.1 & 0.4 \\ 0.9 & 0.3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -1.1 & 0.4 \\ 0 & 0.3 \end{bmatrix}$$

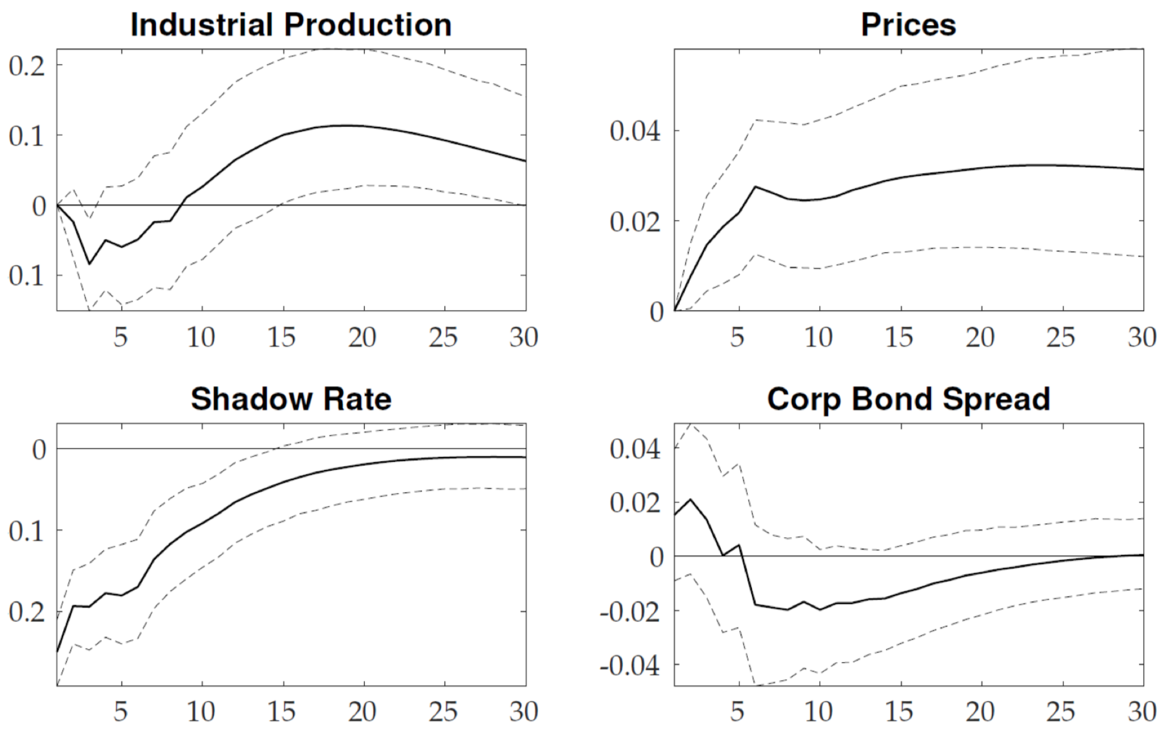
Organizing by shocks, we find that the responses to an interest rate shock is given by $\frac{\partial I_t}{\partial \varepsilon_{It}} = 1$, $\frac{\partial I_{t+1}}{\partial \varepsilon_{It}} = -1.1$, $\frac{\partial S_t}{\partial \varepsilon_{It}} = -3$, $\frac{\partial S_{t+1}}{\partial \varepsilon_{It}} = 0$.
And the responses to a stock price shock are: $\frac{\partial I_t}{\partial \varepsilon_{St}} = 0$, $\frac{\partial I_{t+1}}{\partial \varepsilon_{St}} = 0.4$, $\frac{\partial S_t}{\partial \varepsilon_{St}} = 1$, $\frac{\partial S_{t+1}}{\partial \varepsilon_{St}} = 0.3$.
Stock prices fall contemporaneously by 3 points after a one unit shock in interest rates. In the second period the response of stock prices to the initial shock is zero. Interest rates, by construction, do not respond contemporaneously to stock prices shocks. However, in the second period interest rates rise by 0.4 as a response to the initial shock in stock prices.

4. (15 pts) Figure 3 is taken from a recent European Central Bank working paper by Hafemann and Tillmann (2017). Shadow rate is an interest rate that takes into account unconventional monetary policy. Interpret the figure. Is Corporate Bond Spread ordered after Shadow Rate? Why or why not.

   The figure shows the responses of the four variables to a 0.25 basis points (negative) shock in the shadow rate. Industrial production first falls and then in the medium run it increases and this increase is significant at the 90% confidence level. That, is for he first 10 periods it does not respond much and even falls and then it accelerates to reach a peak of approximately 0.1. Prices increase as a response (and is significant) to the same shock as expected by theory. Hence, in this case there is no "price puzzle". The shadow rate responds to its on shock with some persistence. The shock only seems to die out around period 20. Finally the response of corporate bond spread is insignificant.

   Corporate bond spread is indeed ordered after the shadow rate because it is responding to the shadow rate shock contemporaneously, while industrial production and prices are ordered before shadow rate as they are not affect contemporaneously by the same shock.

Figure 3: **VAR Model with Cholesky Identification**



Notes: Responses to an expansionary monetary policy shock of 25bp obtained from alternative VAR model identified recursively and 90% confidence band.

The End.