1. (25 pts) Consider the following time series model:

$$y_t = 1.35 - 0.4y_{t-1} + 0.05y_{t-2} + \varepsilon_t - 0.5\varepsilon_{t-1} \qquad (1)$$

(a) (5 pts) Identify (1), i.e. what kind of model is (1)? Show whether (1) is stationary or not.

ARMA(2,1) model. The stationary condition in the ARMA model involves in ensuring that the AR component of the ARMA model is stationary. To show stationary, one can either build the characteristic equation and find its roots or use the lag operator to find the inverse characteristic equation. The charactheristic equation here is given by: $\alpha^2 + 0.4\alpha - 0.05$. *The roots are* $\alpha_1 = 0.1$ *and* $\alpha_2 = -0.5$. *Since both roots are real and are in absolute value lower than 1, the ARMA(2,1) is stationary.*

(b) (5 pts) Find the homogeneous solution. Is it convergent? Let $A_1 = A_2 = 1$, what is the type of convergence or divergence (direct or oscillatory)?

The homogeneous solution is given by: $y_t^h = A_1(0.1)^t + A_2(-0.5)^t$. *It is clearly convergent since both roots are below one in absolute terms. Hence, as time increases* y_t^h *converges to zero. With* $A_1 = A_2 = 1$, *the convergence will be oscillatory because of the negative root. At odd time periods* t, y_t^h *is negative and in even periods it will be positive.*

(c) (5 pts) Find the particular solution using the lag operator (no need to find the coefficients). What is the complete solution to (1)?

Applying the lag operator to (1) we find: $y_t = 1.35 - 0.4Ly_t + 0.05L^2y_t + \varepsilon_t - 0.5\varepsilon_{t-1}$. *Solving for* y_t *we can find the particular solution:* $y_t^p = \frac{1.35+\varepsilon_t-0.5\varepsilon_{t-1}}{1+0.4L-0.05L^2}$. *The complete solution is given by* $y_t^h + y_t^p = A_1(0.1)^t + A_2(-0.5)^t + \frac{1.35+\varepsilon_t-0.5\varepsilon_{t-1}}{1+0.4L-0.05L^2}$.

(d) (5 pts) Explain in general terms what is the shape of the ACF and PACF (no need to prove or compute the autocorrelations moments). Find $E[y_t]$, $E_t[y_{t+1}]$ and $E_t[y_{t+2}]$.

The ACF will be affected at lag 1 by the MA component. However, after lag 1 it will exhibit an exponential decay because of the AR component. The PACF will be affect at lag 1 and 2 by the AR component with two lags. However it will also exhibit an exponential decay because of the MA component. $E[y_t] = 1.35/(1 + 0.4 - 0.05) = 1$ *The 1 step ahead forecast* $E_t[y_{t+1}] = 1.35 - 0.4y_t + 0.05y_{t-1} - 0.5\varepsilon_t$ *and the two steps ahead is* $E_t[y_{t+2}] = 1.35 - 0.4E_t[y_{t+1}] + 0.05y_t$

(e) (5 pts) Explain how the different components of (1) affect the forecasts you found in (d).

The MA component makes a one step forecast revision and hence only affects the one step ahead forecasts. Hence, the two step ahead forecast is no longer affect by the MA component. The AR component continues to have an effect on all future forecasts. However since the model is stationary, the forecast will converge eventually to the unconditional mean 1.

2. (20 pts)

(a) (10 pts) Discuss how the Wold Decomposition motivates the use of ARMA models for analysing weakly stationary time series.

The Wold Decomposition shows that any stationary process has a linear $MA(\infty)$ *representation. Hence, it motivates the use of ARMA models has they have a flexible way of achieving the* $MA(\infty)$. *The AR components will ensure that there is an* $MA(\infty)$ *but imposes restrictions on the coefficients of the Wold Decomposition, and the MA components allow flexibility on some of the coefficients. Hence, the ARMA is a sensible way of approximating the* $MA(\infty)$ *of any process without losing too many degrees of freedom.*

(b) (10 pts) Suppose you have the R output given in Figures 1, 2 and 3 (last page). What is the model you identify from the ACF and how many lags would you include? Check the model, is it a good model? (Hint: H_0: No serial correlation)

The exponential decay on the ACF is a strong feature of the AR modes. Moreover, the PACF is insignificant after the second lag. Hence, we identify an AR(2) model. The Ljung-Box statistic is 9.55 with a p-value of 0.6551 so that we do not reject no serial correlation. So the model is good.

3. (40 pts) Suppose we have the following estimated VAR with two variables, money aggregate M and GDP Y:

$$\begin{aligned} M_t &= \underset{(0.01)}{0.3^{***}} Y_{t-1} + \underset{(0.2)}{0.9^{***}} M_{t-1} + e_{2t} \\ Y_t &= \underset{(0.05)}{0.4^{***}} Y_{t-1} + \underset{(0.3)}{0.1} M_{t-1} + e_{1t} \end{aligned}$$

Where the term inside the brackets are the estimates standard errors and the stars represent coefficients that are significant at a 5% confidence level. Also the Cholesky decomposition of the residual variance-covariance matrix is given by:

$$L = \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.4 \end{bmatrix}$$

(a) (10 pts) Does money Granger cause GDP? And does GDP Granger cause money?

To test if money Granger causes GDP is to test whether money helps predicting GDP. We can test that using the two variables VAR. Since the coefficient of M_{t-1} on the GDP equation is not significant, we conclude that money does not Granger causes GDP. However, GDP does Granger cause money.

(b) (10 pts) Find the structural VAR representation with the information given to you (i.e. Find Γ_0 and Γ_1. Hint: no need to inverse matrix).

The Choleski decomposition sets $L = B^{-1}$. Moreover, $A_1 = B^{-1}\Gamma_1 = L\Gamma_1$, and $A_0 = B^{-1}\Gamma_0 = L\Gamma_0$. Since $A_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we have that $\Gamma_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. To find Γ_1, we just need to set up the system:

$L\Gamma_1 = A_1$

$$\begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix} 0.3\Gamma_{11} & 0.3\Gamma_{12} \\ 0.5\Gamma_{11} + 0.4\Gamma_{21} & 0.5\Gamma_{12} + 0.4\Gamma_{22} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.4 \end{bmatrix}$$

We find that $\Gamma_{11} = 3$, $\Gamma_{12} = 1$. Hence, we can also find Γ_{21} using $1.5 + 0.4\Gamma_{21} = 0.1$. We find that $\Gamma_{21} = -7/2$. In the same way we find $\Gamma_{22} = -1/4$
The structural VAR is then given by:

$$\begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.4 \end{bmatrix}^{-1} \begin{bmatrix} M_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1/4 & -7/2 \end{bmatrix} \begin{bmatrix} M_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{Mt} \\ \varepsilon_{Yt} \end{bmatrix}$$

(c) (10 pts) Find the contemporaneous and one step ahead impulse responses using what you found in b). Interpret the results.

Using the $VMA(\infty)$ representation, the contemporaneous IRFs can be found by L. The one-steap ahead can be found by $A_1 L$.

$$\begin{bmatrix} M_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} \varepsilon_{Mt} \\ \varepsilon_{Yt} \end{bmatrix}$$

The contemporaneous IRFs are:

$$\begin{bmatrix} \frac{\partial M_t}{\partial \varepsilon_{Mt}} & \frac{\partial M_t}{\partial \varepsilon_{Yt}} \\ \frac{\partial Y_t}{\partial \varepsilon_{Mt}} & \frac{\partial Y_t}{\partial \varepsilon_{Yt}} \end{bmatrix} = \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.4 \end{bmatrix}$$

The one-step ahead $A_1 L$ will be given by:

$$\begin{bmatrix} \frac{\partial M_{t+1}}{\partial \varepsilon_{Mt}} & \frac{\partial M_{t+1}}{\partial \varepsilon_{Yt}} \\ \frac{\partial Y_{t+1}}{\partial \varepsilon_{Mt}} & \frac{\partial Y_{t+1}}{\partial \varepsilon_{Yt}} \end{bmatrix} = \begin{bmatrix} 0.42 & 0.12 \\ 0.23 & 0.16 \end{bmatrix}$$

A one unit increase in money has a 0.3 immediate increase effect on money and a 0.5 immediate increase effect on GDP. The scale of each variable is not known here. The same shock today will propagate and will have a bigger effect on money one step ahead with a 0.42 increase, while the response for GDP wrt the same shock is smaller with an increase of 0.23. A unit shock to GDP has no immediate affect on money by construction and it has an effect on GDP with 0.4 increase. Next period though, the same GDP shock will have a positive effect on money with 0.12 increase. And the GDP effect disappears quickly as the increase next period is only 0.16.

(d) (10 pts) Interpret the forecast error variance decomposition in Table 1. Discuss how the information contained in the FEVD and the IRFs complement each other.

The FEVD shows that the unexplained portion of money variation is totally explained by money at time 0 by construction (Cholesky decomposition) and that after 20 periods the its own shocks explain 50% of the variation while shocks to GDP explain also 50%. This shows that shocks to GDP are important in explaining unexpected movements of money. On the other hand, GDP unexplained movements are mainly driven by GDP itself as money explains very little of the total variation. The FEVD and IRFs complement each other because with the IRF, we are looking at a simulation whereby we look at how the variables in the system react to a single shock at a time. While the FEVD shows the relative importance of the shocks in explaining variation in the variables when all shocks happen at the same time. For instance, in this example we find with the IRF that GDP reacts strongly to money shocks holding all else constant. The FEVD tells us that that IRF is not very relevant since when all shocks happen at the same time, GDP reacts only to shocks in itself. In other words, money shocks are relatively unimportant in explaining GDP volatility.

4. (15 pts) Consider the following model:

$$y_t = 0.9y_{t-1} + \varepsilon_t$$

(a) (7.5 pts) Show how to build a simple Dickey-Fuller unit root test for the model above. Suppose $\tau = -5.2$ and $\tau^* = -4$ with 5% confidence level, is y_t stationary?

We can subtract y_{t-1} on both sides:

$\Delta y_t = -0.1y_{t-1} + \varepsilon_t$

The Dickey-Fuller tests $H_0 : \gamma = 0$. Here, $\gamma = -0.1$ In this, since $\tau < \tau^*$ we reject the null hypothesis and conclude that we reject the unit root hypothesis. Hence, we reject that the process is not stationary. y_t looks to be stationary.

(b) (7.5 pts) What is the assumption on the Dicky-Fuller test? How can we deal with the situation in which such assumption is violated?

The Dickey-Fuller assumes that ε_t is stationary. If this assumption is violated, the DF test is no longer valid. One way to deal with this is to include lags on the variable differences. We include as many as need to ensure that the ε_t is stationary. This is called the Augmented Dickey-Fuller test.

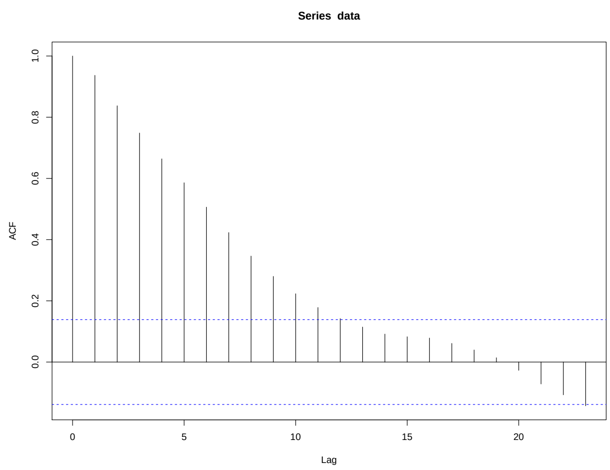

Figure 1: ACF

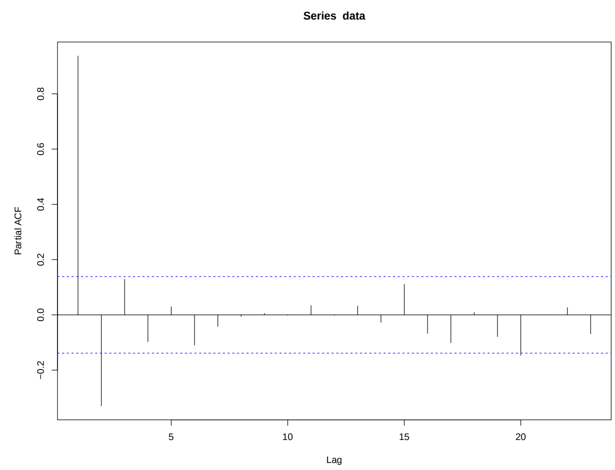

Figure 2: PACF

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        Box-Ljung test

data:  arima1$residuals
X-squared = 9.5528, df = 12, p-value = 0.6551
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Figure 3: Ljung-Box test on the fittet model

Money - M		
Steps	Y (%)	M (%)
0	0	100
5	20	80
10	30	70
20	50	50
GDP - Y		
Steps	Y (%)	M (%)
0	95	5
5	95	5
10	94	6
20	90	10

Table 1: Hypothetical FEVD for Question 3 part d)

The End.