Exercises Week 1

1. Difference Equations (Revision)

1. Consider the AR(1) model (stationary):

$$y_t = \alpha y_{t-1} + \epsilon_t \tag{1}$$

a) Show that $E(Y_t) = 0$

Solution:

Let L be the lag operator such that $Ly_t = y_{t-1}$. Then:

$$(1 - \alpha L)y_t = \epsilon_t \Leftrightarrow$$
$$y_t = \frac{1}{1 - \alpha L}\epsilon_t$$

Because $\alpha < 1$ since the model is stationary, we can use the geometric sum formula such that:

$$y_t = \sum_{i=0}^{\infty} \alpha^i \epsilon_{t-i}$$

This is the MA(∞) representation of the stationary AR(1) process. Now we can take the expected value:

$$E(y_t) = E(\sum_{i=0}^{\infty} \alpha^i \epsilon_{t-i}) = E(\epsilon_t) = 0$$

b) Calculate the variance:

Solution:

Let's start from the MA(∞) specification and use the rules of the variance

$$Var(y_t) = Var(\sum_{i=0}^{\infty} \alpha^i \epsilon_{t-i}) = \frac{1}{1 - \alpha^2} Var(\epsilon_t) = \frac{\sigma_e^2}{1 - \alpha^2}$$

2. Consider the AR(2) model:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \epsilon_t \tag{2}$$

a) Find the equilibrium solution $y*$.

Solution:

By substitution:

$$y^* = \alpha_0 + \alpha_1 y^* + \alpha_2 y^* + \epsilon_t \Leftrightarrow$$
$$y^* = \frac{\alpha_0}{(1 - \alpha_1 - \alpha_2)}$$

b) Show the characteristic polynomial and describe what are the stability conditions.

Solution:

Write the model using the Lag Operator:

$$y_t = \alpha_0 + \alpha_1 L y_t + \alpha_2 L^2 y_t + \epsilon_t \Leftrightarrow$$
$$(1 - \alpha_1 L - \alpha_2 L^2) y_t = \alpha_0 + \epsilon_t$$

Factorize the characteristic polynomial and find the roots:

$$(1 - \alpha_1 L - \alpha_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L) \Leftrightarrow$$
$$(1 - \alpha_1 L - \alpha_2 L^2)^{(-1)} = (1 - \lambda_1 L)^{(-1)}(1 - \lambda_2 L)^{(-1)}$$

The roots of characteristic polynomial $\lambda^2 - \alpha_1 \lambda - \alpha_2 = 0$:

$$\lambda_1, \lambda_2 = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2}$$

- If $\lambda_1 \neq \lambda_2$ real: Stability Condition: $|\lambda_1| < 1$ and $|\lambda_2| < 1$
- If $\lambda_1 \neq \lambda_2$ real: Stability Condition: $|\lambda_1| < 1$ and $|\lambda_2| < 1$
- If $\lambda_1 = a + bi$, $\lambda_2 = a - bi$ complex: Stability Condition: $|r| = r < 1$

2. Forecasting

1. Consider the European area economic sentiment indicator (**SENTIMENT**) series in Figure 1. To make some policy decision it was necessary to provide

Figure 1: European area economic sentiment indicator

one-step ahead forecasts for the Sentiment Indicator. To help choose the model to be used in the forecasting exercise the period from 2018:02 to 2020:02 was chosen for evaluation purposes. Two models where considered: i) Model A - a simple AR(4) and Model B - a simple trend model ($Sentiment_t = a + bt + e_t$).

a) Given the following results, which model would you chose. Justify.

	AR(4)	Trend Model
Mean Error	-9.0148	-39.513
Mean Squared Error	71.134	74.492
Root Mean Squared Error	8.4341	8.6309
Mean Absolute Error	57.733	66.642
Mean Percentage Error	131.16	212.94

Solution:
Model - AR(4) as it presents the lowest values for all forecast measures provided

b) In addition given the results in Table below indicate in detail how you would compute the Diebold-Mariano test.
Solution:
To compute the Diabold-Mariano test we need to do the following:
   i) Compute the forecast errors for Model A and Model B, $\hat{\epsilon_t^A}$ and $\hat{\epsilon_t^B}$, respectively, where $\hat{\epsilon_t^K} = Model\ K_t - Sentiment_t$, $K = A, B$

Date	Sentiment	Model A	Model B
2018:2	-8.3	68.0	40.3
2018:3	-37.0	-29.6	36.9
2018:4	-62.4	-15.5	32.2
2019:1	-40.0	-46.3	26.4
2019:2	-17.3	-3.0	22.0
2019:3	-86.3	8.1	19.0
2019:4	-13.3	-88.9	12.7
2020:1	-13.5	47.6	10.1
2020:2	129.8	-7.6	7.7

Table 1: Forecast Results

ii) Compute either

$$\hat{d}_t = (\hat{\epsilon_t^A})^2 - (\hat{\epsilon_t^B})^2 \quad or$$
$$\hat{d}_t = |\hat{\epsilon_t^A}| - |\hat{\epsilon_t^B}|$$

iii) Compute the mean of the $\hat{d}_t$:

$$\bar{d} = \frac{1}{n}\sum_{t=1}^{n} \hat{d}_t$$

iv) Compute the DM test

$$DM = \frac{\bar{d}}{\sqrt{Var(\bar{d})}}$$

Note that $Var(\bar{d})$ needs to be estimated using the **Newey-West HAC** estimator as the forecast generates autocorrelation.