NOVA SCHOOL OF BUSINESS & ECONOMICS

MACROECONOMETRICS

Master in Economics

Difference Equations
Part II – Doing the Math

Difference Equations

Topics

Linear difference equations:

- **1st order**
- **2nd order**
- ***p*th order**

1st order Difference Equations

1. Finding the solution given an initial condition
2. Equilibrium solution
3. Impulse response function
4. Stability condition
5. Finding a solution without an initial condition
6. Finding a solution using the Lag operator

1st order Difference Equations

$$y_t = a_0 + a_1 \, y_{t\text{-}1} + \varepsilon_t \quad \text{for all } t$$

Finding a solution for $\{y_t\}$
as a function of the forcing process $\{\varepsilon_t\}$

We will look at three approaches:

1. Finding a solution given an initial condition

5. Finding a solution without an initial condition

6. Finding a solution using the Lag operator

1st order Difference Equations

1. Finding the solution given an initial condition

- We start at time t-1
- Assume y_{t-1} is known
- Iterate forward from t-1 to t , t+1 , ... , t+n, ...

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$$

$$y_{t+1} = a_0 + a_1 y_t + \varepsilon_{t+1}$$

$$= a_0 + a_1 (a_0 + a_1 y_{t-1} + \varepsilon_t) + \varepsilon_{t+1}$$

$$= (a_0 + a_1 a_0) + a_1^2 y_{t-1} + (a_1 \varepsilon_t + \varepsilon_{t+1})$$

1st order Difference Equations

1. Finding the solution given an initial condition

...after $n+1$ iterations (from t to $t+n$):

$$y_{t+n} = a_0\,\underbrace{(1+a_1+a_1^2+\cdots+a_1^n)}_{n+1\text{ terms}}$$

$$+a_1^{n+1}\,y_{t-1}$$

$$+\overbrace{(a_1^n\varepsilon_t+a_1^{n-1}\varepsilon_{t+1}+\cdots+a_1\varepsilon_{t+n-1}+\varepsilon_{t+n})}^{n+1\text{ terms}}$$

1st order Difference Equations

2. Equilibrium solution

$$y_t = a_0 + a_1\, y_{t-1} + \varepsilon_t$$

Is there an equilibrium solution?

Meaning: a value for y, call it y^*, such that if :

- $y_{t-1} = y^*$
- and no future shocks: $\varepsilon_t = \varepsilon_{t+1} = \cdots = \varepsilon_{t+n} = \cdots = 0$

then $y_t = y_{t+1} = \cdots = y_{t+n} = \cdots = y^*$.

Let's find it: $y^* = a_0 + a_1\, y^* \Rightarrow$ $$y^* = a_0 / (1 - a_1)$$

Note: $a_0 = (1 - a_1)\, y^*$

1st order Difference Equations

2. Equilibrium solution

Note: We can rewrite

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$$

as

$$a_0 = (1 - a_1)\, y^*$$

$$y_t = y^* + a_1 (y_{t-1} - y^*) + \varepsilon_t$$

As before, find solution by iterating forward from y_{t-1} :

... and after $n+1$ iterations (from t to $t+n$) we get:

$$y_{t+n} = y^* + a_1^{n+1} (y_{t-1} - y^*) + (a_1^n \varepsilon_t + a_1^{n-1} \varepsilon_{t+1} + \cdots + a_1 \varepsilon_{t+n-1} + \varepsilon_{t+n})$$

1st order Difference Equations

3. Impulse response function

$$y_{t+n} = y^* + a_1^{n+1}(y_{t-1} - y^*) + (a_1^n \varepsilon_t + a_1^{n-1}\varepsilon_{t+1} + \cdots + a_1 \varepsilon_{t+n-1} + \varepsilon_{t+n})$$

What are the impacts of a **TRANSITORY** shock ε_t

(meaning: $\varepsilon_t = 1$, $\varepsilon_{t+1} = \varepsilon_{t+2} = \ldots = 0$)

on the present and future values of y_t ? (multipliers ≡ IRF)

$$\partial y_{t+n} / \partial \varepsilon_t = \begin{cases} 1 & \text{for } n = 0 \\ a_1^n & \text{for } n = 1, 2, 3, \ldots \end{cases}$$

1st order Difference Equations

3. Impulse response function

Note:

$$\partial y_{t+n} / \partial \varepsilon_t = \partial y_t / \partial \varepsilon_{t-n}$$

$$= \begin{cases} 1 & \text{for } n = 0 \\ a_1{}^n & \text{for } n = 1, 2, 3, \ldots \end{cases}$$

1st order Difference Equations
4. Stability condition

How does the IRF look like for different $\boldsymbol{a_1}$?

$a_1 > 1$ → explosive

$a_1 = 1$ → permanent impact

$0 < a_1 < 1$ → dies out exponentially

$a_1 = 0$ → impact only when shock occurs, not in future

$-1 < a_1 < 0$ → dies out expon. & oscillates +/- every period

$a_1 = -1$ → permanent impact & oscillates +/- every period

$a_1 < -1$ → explosive & oscillates +/- every period

1st order Difference Equations

4. Stability condition

Some simulations.

In all these cases:

- initial condition at time 0 : $y_0 = 1$

- ε_t are random shocks

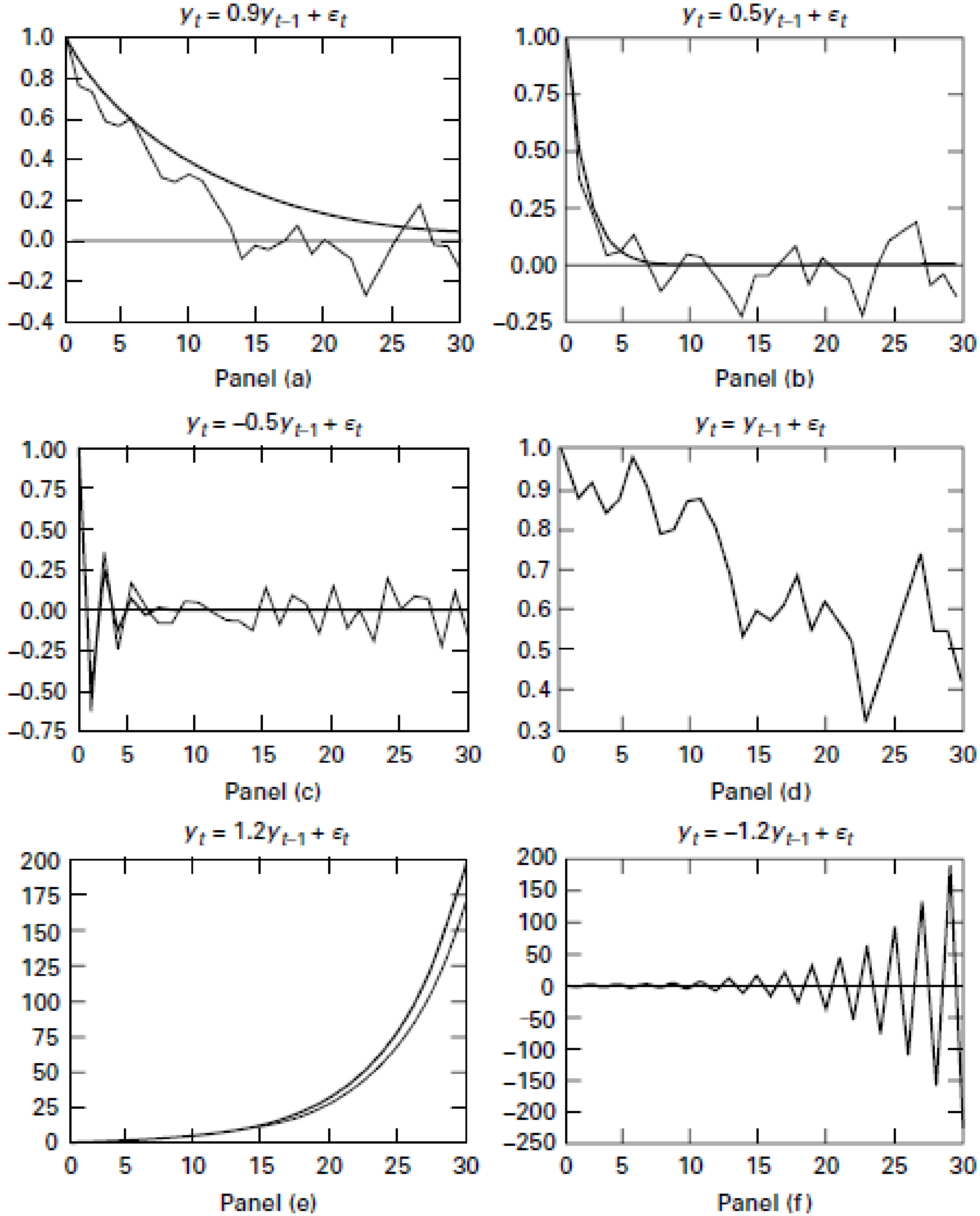

FIGURE 1.2 Convergent and Nonconvergent Sequences

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1st order Difference Equations

4. Stability condition

Simulation in Gretl: sim_diff_eq_1.inp

```
# Simulate a 1st order difference equation:
#      y(t) = a0 + a1*y(t-1) + epsilon(t)
# epsilon: random shocks
# y: the solution
# y_zeroshocks: solution without shocks

# Create data set with annual data, 1900-1950
nulldata 51
setobs 1 1900 --time-series

# Generate Normal random shocks with
# some standard deviation and zero mean
scalar epsilon_sd=0.1
series epsilon=normal(0,epsilon_sd)
# Set a0 and a1
scalar a0=0
scalar a1=0.9
```

```
# Initialize the two series: y , y_zeroshocks
series y = 0
series y_zeroshocks = 0

# Initial condition
smpl 1900 1900
y = 1
y_zeroshocks = 1
# Iterate forward until the end of the sample
smpl 1901 1950
y = a0 + a1*y(-1) + epsilon
y_zeroshocks = a0 + a1*y_zeroshocks(-1)

# Finally, reset to the full sample
smpl 1900 1950

# Graph the simulated series
gnuplot y y_zeroshocks --time-series --with-lines --
output=display
```

1st order Difference Equations

4. Stability condition

$$|a_1| < 1$$

1st order Difference Equations

Back to 3. Impulse response function

$$y_{t+n} = y^* + a_1^{n+1}(y_{t-1} - y^*) + (a_1^n \varepsilon_t + a_1^{n-1}\varepsilon_{t+1} + \cdots + a_1 \varepsilon_{t+n-1} + \varepsilon_{t+n})$$

What are the impacts of a **PERMANENT** shock ?

This means that ε_t, ε_{t+1}, ε_{t+2}, ... , all increase by 1 unit.

Impact on y_{t+n} is $a_1^n + a_1^{n-1} + \cdots + a_1 + 1$

Impact of a permanent shock is the cumulative IRF !

1st order Difference Equations
Back to 3. Impulse response function

Suppose stability condition holds: $|a_1| < 1$

Long-run impact on y of a:

- transitory shock in ε is equal to zero
- permanent shock in ε is given by $1/(1-a_1)$

1st order Difference Equations

5. Finding a solution without an initial condition

- We start at time t
- Iterate backward:

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$$

$$y_t = a_0 + a_1 (a_0 + a_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$

$$= (a_0 + a_1 a_0) + a_1^2 y_{t-2} + (a_1 \varepsilon_{t-1} + \varepsilon_t)$$

$$y_t = (a_0 + a_1 a_0 + a_1^2 a_0 + \cdots) + a_1^\infty y_{t-\infty} + (\varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2} + \cdots)$$

Note: just to simplify, the notation is a bit sloppy: one should use limits.

1st order Difference Equations

5. Finding a solution without an initial condition

Assumptions:

- $\{ y_t \}$ is bounded, meaning $| y_t | < \infty$ for all t
- Stability condition holds: $| a_1 | < 1$

We have found a solution:

$$y_t = y^* + (\varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2} + \cdots)$$

where: $y^* = a_0 / (1 - a_1)$

1st order Difference Equations

5. Finding a solution without an initial condition

A Solution:

$$y_t = y^* + (\varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2} + \cdots)$$

Note: Because there is no initial condition,

there are actually other possible solutions, such as:

$$y_t = c\, a_1^t + y^* + (\varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2} + \cdots)$$

where c is some constant.

But also note that this solution is unbounded since

ca_1^t diverges when $t \to -\infty$

unless $c = 0$.

1st order Difference Equations

6. Finding a solution using the Lag operator

The Lag operator: $L\, y_t = y_{t\text{-}1}$ & $L\, c = c$

Rewrite:

$$y_t = a_0 + a_1\, y_{t\text{-}1} + \varepsilon_t$$

as:

$$y_t = a_0 + a_1\, Ly_t + \varepsilon_t$$

$$\Leftrightarrow$$

$$(\, 1 - a_1\, L)\, y_t = a_0 + \varepsilon_t$$

1st order Difference Equations

6. Finding a solution using the Lag operator

Can we solve:

$$(1 - a_1 L)\, y_t = a_0 + \varepsilon_t$$

as:

$$y_t = (1 - a_1 L)^{-1} (a_0 + \varepsilon_t) \quad ?$$

Answer: Yes, if these assumptions are valid:

- $\{ y_t \}$ is bounded,
- Stability condition: $| a_1 | < 1$.

In this case:

$$(1 - a_1 L)^{-1} \equiv (1 + a_1 L + a_1^2 L^2 + \cdots)$$

1st order Difference Equations

6. Finding a solution using the Lag operator

Doing the derivations we have:

$$(1 - a_1 L)\, y_t = a_0 + \varepsilon_t$$

$$\Leftrightarrow \quad y_t = (1 - a_1 L)^{-1} (a_0 + \varepsilon_t)$$

$$\Leftrightarrow \quad y_t = (1 + a_1 L + a_1^2 L^2 + \cdots)(a_0 + \varepsilon_t)$$

$$\Leftrightarrow \quad y_t = (1 + a_1 L + a_1^2 L^2 + \cdots)\, a_0 + (1 + a_1 L + a_1^2 L^2 + \cdots)\, \varepsilon_t$$

$$\Leftrightarrow \quad y_t = (a_0 + a_1 a_0 + a_1^2 a_0 + \cdots) + (\varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2} + \cdots)$$

$$\Leftrightarrow \quad y_t = y^* + (\varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2} + \cdots)$$

2nd order Difference Equations

1. **Finding the solution given initial conditions**
2. **Equilibrium solution**
3. **Using the Lag operator**
4. **IRF and stability conditions**

2nd order Difference Equations

1. Finding the solution given initial conditions

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \varepsilon_t$$

Iterating forward→ Need 2 initial conditions

For instance, if we know values of $\boldsymbol{y_{-1}}$ and $\boldsymbol{y_0}$ then:

$$y_1 = a_0 + a_1 y_0 + a_2 y_{-1} + \varepsilon_1$$

$$y_2 = a_0 + a_1 y_1 + a_2 y_0 + \varepsilon_2$$

$$= a_0 + a_1(a_0 + a_1 y_0 + a_2 y_{-1} + \varepsilon_1) + a_2 y_0 + \varepsilon_2$$

etc,...

Examples: use Gretl program sim_dif_eq_2.inp

2nd order Difference Equations

2. Equilibrium solution

$$y_t = a_0 + a_1\, y_{t\text{-}1} + a_2\, y_{t\text{-}2} + \varepsilon_t$$

Is there an equilibrium solution?

Meaning: a value for y, call it y^*, such that if :

- $y_{t\text{-}1} = y^*$
- and no future shocks: $\varepsilon_t = \varepsilon_{t+1} = \cdots = \varepsilon_{t+n} = \cdots = 0$

then $y_t = y_{t+1} = \cdots = y_{t+n} = \cdots = y^*$.

Let's find it: $y^* = a_0 + a_1\, y^* + a_2\, y^* \Rightarrow$

$$y^* = a_0 / (1 - a_1 - a_2)$$

2nd order Difference Equations

3. Using the Lag operator

Let's use the lag operator:

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \varepsilon_t$$

$$\Leftrightarrow (1 - a_1 L - a_2 L^2)\, y_t = a_0 + \varepsilon_t$$

$$\Leftrightarrow y_t = (1 - a_1 L - a_2 L^2)^{-1} (a_0 + \varepsilon_t)$$

Is it possible? What does it mean?

2nd order Difference Equations
3. Using the Lag operator

Suppose we can factorize:

$$(1-a_1L-a_2\,L^2) = (1-\lambda_1L)\,(1-\lambda_2L)$$

so that

$$(1-a_1L-a_2\,L^2)^{-1} = (1-\lambda_1L)^{-1}\,(1-\lambda_2L)^{-1}$$

Note: This result is valid only when Stability Conditions are ok (we'll check this later)

2nd order Difference Equations

3. Using the Lag operator

It follows that:

$$y_t = (1 - a_1 L - a_2 L^2)^{-1} a_0 + (1 - a_1 L - a_2 L^2)^{-1} \varepsilon_t$$

$$\Leftrightarrow y_t = a_0/(1 - a_1 - a_2) + (1 - \lambda_2 L)^{-1} (1 - \lambda_1 L)^{-1} \varepsilon_t$$

$$\Leftrightarrow y_t = y^* + (1 - \lambda_2 L)^{-1} (\varepsilon_t + \lambda_1 \varepsilon_{t-1} + \lambda_1^2 \varepsilon_{t-2} + \cdots)$$

2nd order Difference Equations

3. Using the Lag operator

$$
\begin{aligned}
y_t &= y^* \\
&\quad + (1-\lambda_2 L)^{-1}(\varepsilon_t + \lambda_1 \varepsilon_{t-1} + \lambda_1^2 \varepsilon_{t-2} + \cdots) \\
\Leftrightarrow y_t &= y^* \\
&\quad + (\varepsilon_t + \lambda_2 \varepsilon_{t-1} + \lambda_2^2 \varepsilon_{t-2} + \cdots) \\
&\quad + (\lambda_1 \varepsilon_{t-1} + \lambda_2 \lambda_1 \varepsilon_{t-2} + \lambda_2^2 \lambda_1 \varepsilon_{t-3} + \cdots) \\
&\quad + (\lambda_1^2 \varepsilon_{t-2} + \lambda_2 \lambda_1^2 \varepsilon_{t-3} + \lambda_2^2 \lambda_1^2 \varepsilon_{t-4} + \cdots) \\
&\quad + \cdots
\end{aligned}
$$

2nd order Difference Equations

3. Using the Lag operator

$$\Leftrightarrow y_t = y^* + (\varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \psi_3 \varepsilon_{t-3} + \cdots)$$

$$\Leftrightarrow y_t = y^* + \psi(L)\varepsilon_t$$

where

$$\psi(L) = 1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \cdots$$

2nd order Difference Equations

3. Using the Lag operator

Summary

$$(1 - a_1 L - a_2 L^2)\, y_t = a_0 + \varepsilon_t$$

$$\Leftrightarrow \quad a(L)\, y_t = a_0 + \varepsilon_t$$

$$\Leftrightarrow \quad y_t = a(L)^{-1} a_0 + a(L)^{-1} \varepsilon_t$$

$$\Leftrightarrow \quad y_t = y^* + \psi(L)\, \varepsilon_t$$

2nd order Difference Equations

4. IRF and stability conditions

$$y_t = y^* + (\varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \psi_3 \varepsilon_{t-3} + \cdots)$$

Impulse Response Function

What are the impacts of a **TRANSITORY** shock ε_t on the present and future values of y_t ? (multipliers ≡ IRF)

$$\partial y_{t+n} / \partial \varepsilon_t = \begin{cases} 1 & \text{for } n = 0 \\ \psi_n & \text{for } n = 1, 2, 3, \ldots \end{cases}$$

2nd order Difference Equations

4. IRF and stability conditions

How to factorize:

$$(1 - a_1 L - a_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L) \quad ?$$

Find roots of characteristic polynomial:

$$i.e., \text{ solve: } \quad \lambda^2 - a_1 \lambda - a_2 = 0$$

2nd order Difference Equations
4. IRF and stability conditions

Find solutions of $\lambda^2 - a_1\lambda - a_2 = 0$

$$\boxed{\lambda_1, \lambda_2 = \frac{a_1 \pm \sqrt{a_1^{\,2} + 4a_2}}{2}}$$

2nd order Difference Equations

4. IRF and stability conditions

Case 1. $\lambda_1 \neq \lambda_2$ real

Case 2. $\lambda_1 = \lambda_2$ real

Case 3. λ_1 and λ_2 complex conjugate,

$$\lambda_1 = a + b\,i \;\;, \lambda_2 = a - b\,i$$

2nd order Difference Equations

4. IRF and stability conditions

Case 1. $\lambda_1 \neq \lambda_2$ real

Impulse Response Function

$$\psi_n = \partial y_{t+n} / \partial \varepsilon_t = c_1 \lambda_1{}^n + c_2 \lambda_2{}^n$$

where c_1 and c_2 are some constants

2nd order Difference Equations

4. IRF and stability conditions

Case 1. $\lambda_1 \neq \lambda_2$ real

Impulse Response Function

$$\psi_n = \partial\, y_{t+n} / \partial\, \varepsilon_t = c_1\, \lambda_1{}^n + c_2\, \lambda_2{}^n$$

Stability Conditions: $|\lambda_1| < 1$ and $|\lambda_2| < 1$

2nd order Difference Equations

4. IRF and stability conditions

Case 2. $\lambda_1 = \lambda_2$ real

Impulse Response Function

$$\psi_n = \partial\, y_{t+n} / \partial\, \varepsilon_t = c_1\, \lambda_1^{\,n} + c_2\, n\lambda_2^{\,n\text{-}1}$$

where c_1 and c_2 are some constants

2nd order Difference Equations

4. IRF and stability conditions

Case 2. $\lambda_1 = \lambda_2$ real

Impulse Response Function

$$\psi_n = \partial y_{t+n} / \partial \varepsilon_t = c_1 \lambda_1^n + c_2 n\lambda_2^{n-1}$$

Stability Conditions: $|\lambda_1| < 1$ and $|\lambda_2| < 1$

2nd order Difference Equations

4. IRF and stability conditions

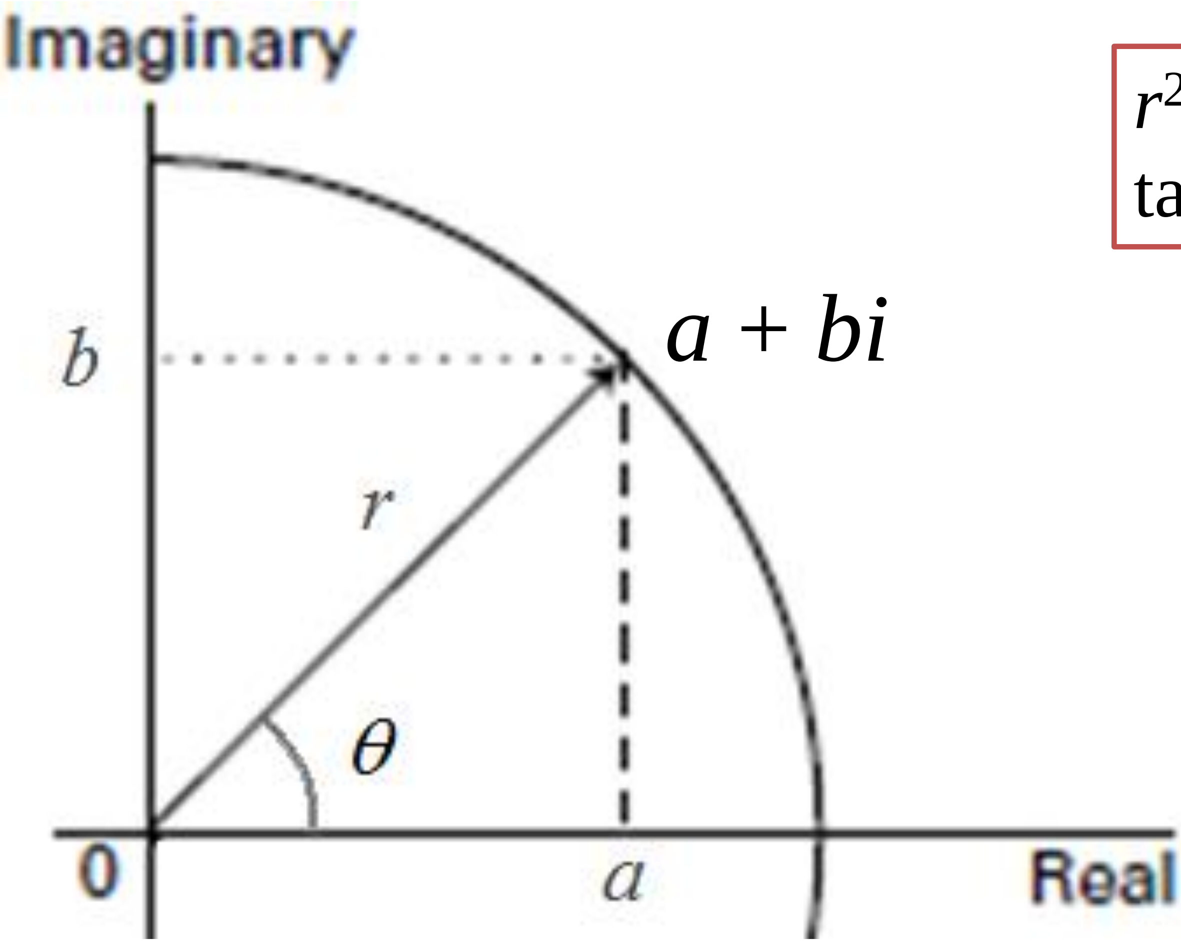
Case 3. $\lambda_1 = a + b\,i \;,\; \lambda_2 = a - b\,i$

Before showing the IRF

we review the "complex numbers"

2nd order Difference Equations

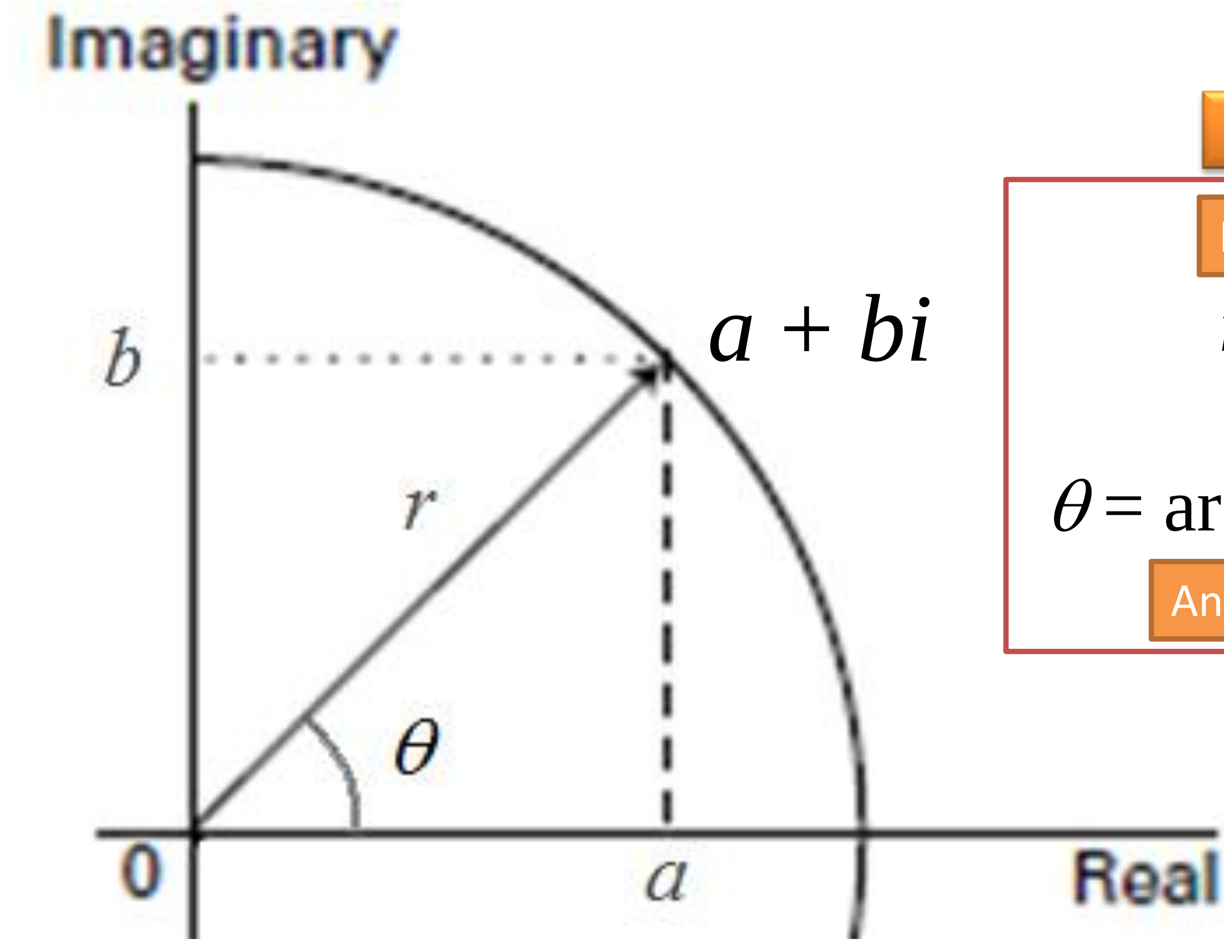
4. IRF and stability conditions



$$r^2 = a^2 + b^2$$
$$\tan(\theta) = b / a$$

2nd order Difference Equations

4. IRF and stability conditions



POLAR COORDINATES

Modulus or amplitude

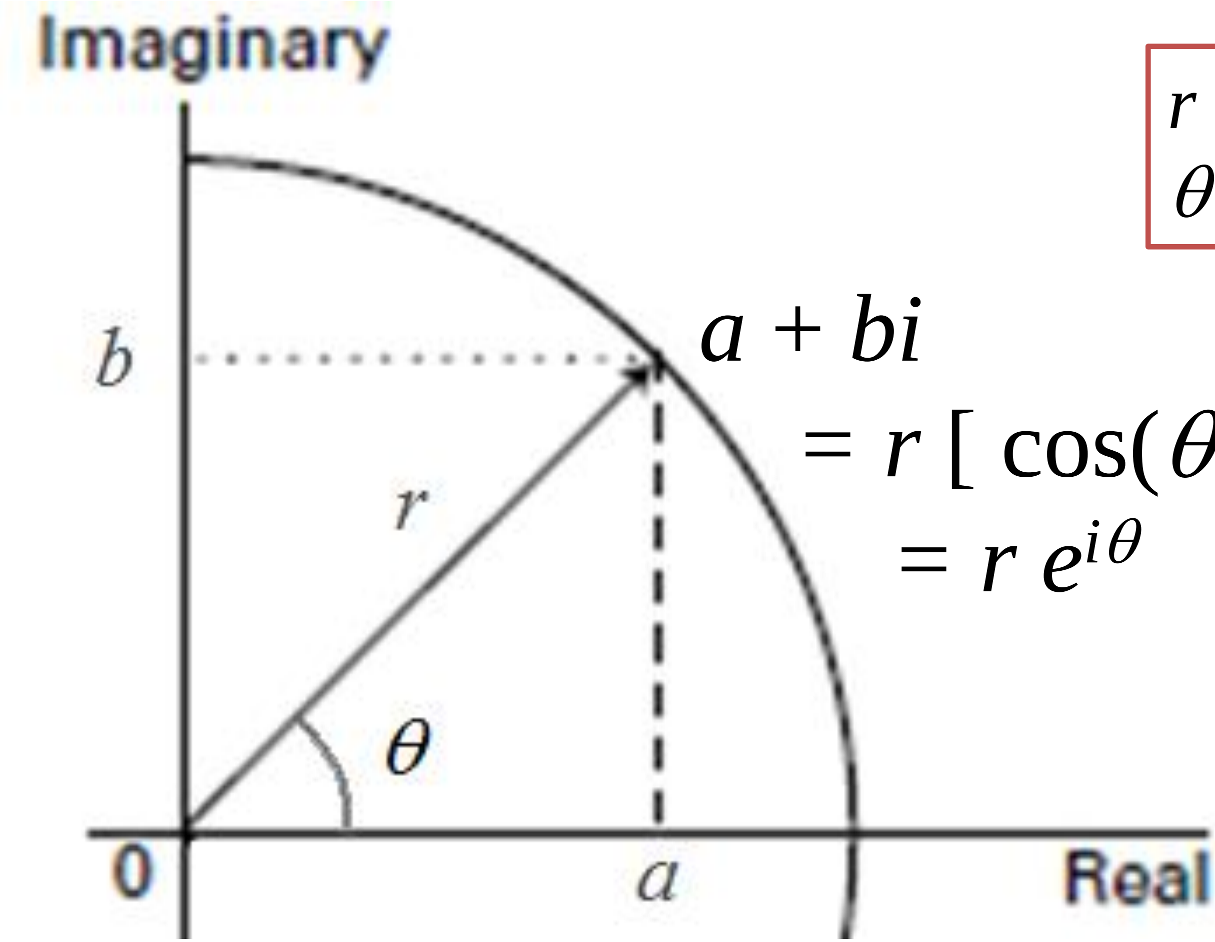
$$r = [a^2 + b^2]^{1/2}$$

$$\theta = \arctan(b/a) = \arg(b/a)$$

Angle , argument, or phase

2nd order Difference Equations

4. IRF and stability conditions



$$r = [a^2 + b^2]^{1/2}$$
$$\theta = \arg\,(b \,/\, a\,)$$

$$a + bi$$
$$= r\,[\,\cos(\theta\,) + i\,\sin\,(\,\theta\,)\,]$$
$$= r\,e^{i\theta}$$

2nd order Difference Equations

4. IRF and stability conditions

Case 3. $\lambda_1 = a + b\,i \;,\; \lambda_2 = a - b\,i$

Impulse Response Function

$$\psi_n = \partial\, y_{t+n} / \partial\, \varepsilon_t = c_1 \lambda_1{}^n + c_2 \lambda_2{}^n$$
$$= \beta_1 r^n \cos(\theta n + \beta_2)$$

where c_1 , c_2 , β_1, and β_2 are some constants

2nd order Difference Equations

4. IRF and stability conditions

Case 3. $\lambda_1 = a + b\,i \;,\; \lambda_2 = a - b\,i$

Impulse Response Function

$$\psi_n = \partial\, y_{t+n} / \partial\, \varepsilon_t = c_1\, \lambda_1{}^n + c_2\, \lambda_2{}^n$$
$$= \beta_1\, r^n\; \cos(\,\theta\, n + \beta_2\,)$$

Stability Condition: $|r| = r < 1$

2nd order Difference Equations

4. IRF and stability conditions

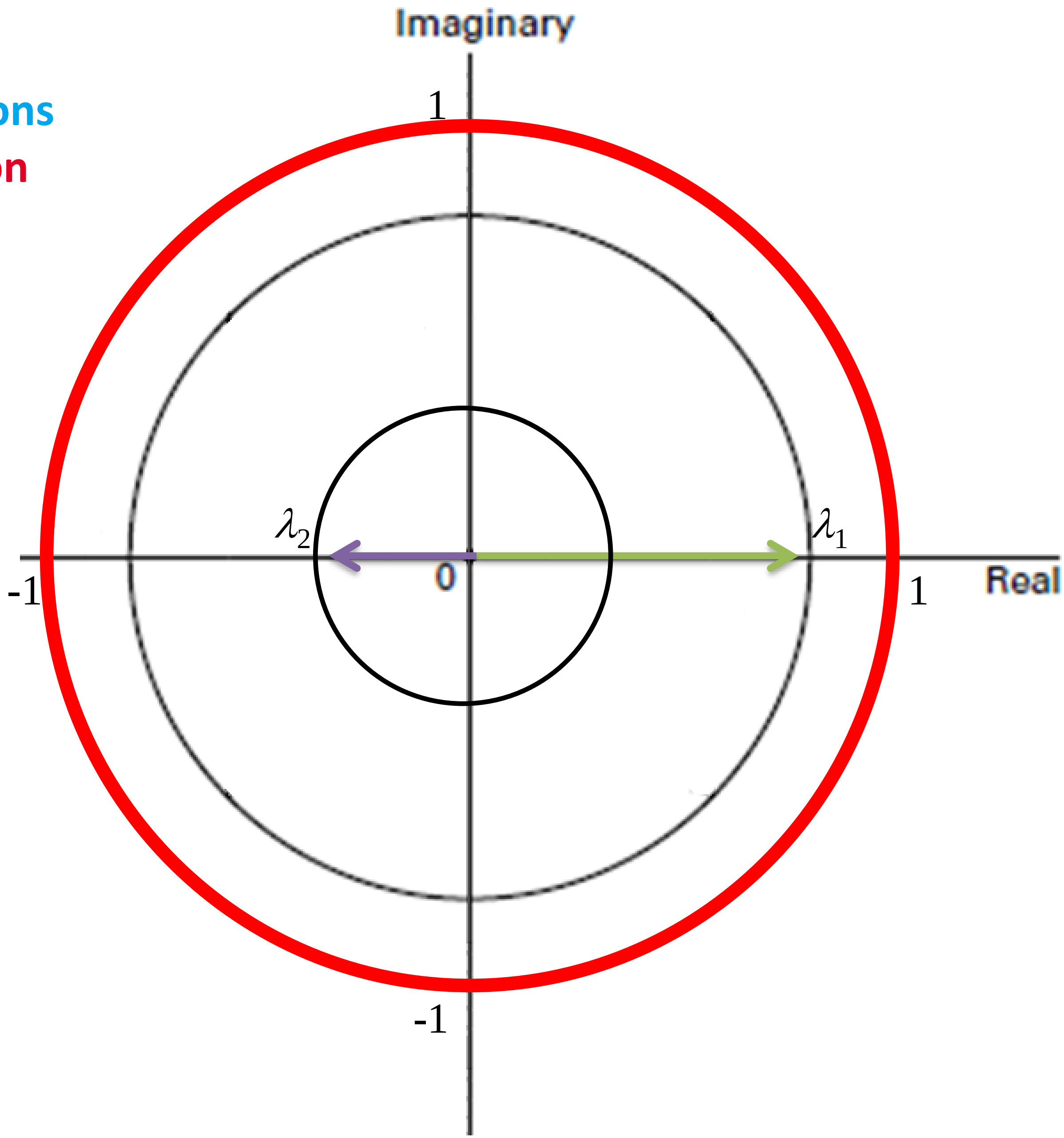
Case 3. $\lambda_1 = a + b\,i \;, \lambda_2 = a - b\,i$

Impulse Response Function

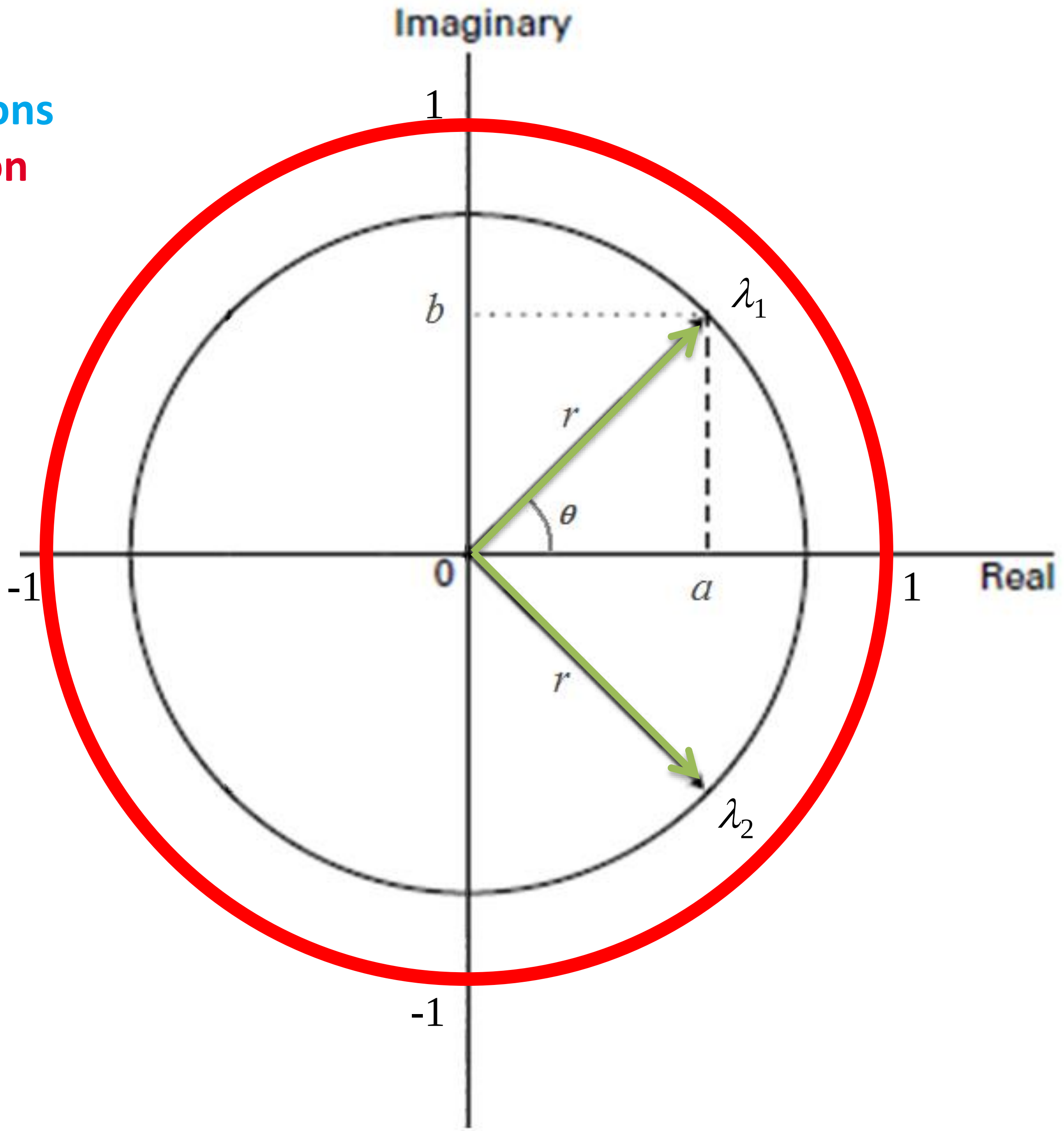
$$\psi_n = \partial\, y_{t+n} / \partial\, \varepsilon_t = c_1\, \lambda_1{}^n + c_2\, \lambda_2{}^n$$
$$= \beta_1\, r^n \cos(\theta\, n + \beta_2)$$

Generates sinusoidal cycles
with periodicity $2\pi/\theta$

Stability Condition for all cases 1-3: Characteristic roots λ_1 and λ_2 have to be inside the unit circle



Stability Condition for all cases 1-3: Characteristic roots λ_1 and λ_2 have to be inside the unit circle



p^{th} order Difference Equations

1. **Finding the solution given initial conditions**

2. **Equilibrium solution**

3. **Using the Lag operator**

4. **IRF and stability conditions**

p^{th} order Difference Equations

1. Finding the solution given initial conditions

$$y_t = a_0 + a_1 y_{t-1} + \cdots + a_p y_{t-p} + \varepsilon_t$$

Iterating forward→ Need *p* initial conditions

Example for p=3: Gretl program sim_dif_eq_3.inp

p^{th} order Difference Equations
2. Equilibrium solution

$$y_t = a_0 + a_1\, y_{t\text{-}1} + \cdots + a_p\, y_{t\text{-p}} + \varepsilon_t$$

Equilibrium solution:

$$y^* = a_0 \,/\, (1 - a_1 - \cdots - a_p\,)$$

p^{th} order Difference Equations

3. Using the Lag operator

Assuming Stability Condition:

$$(1 - a_1 L - \cdots - a_p L^p)\, y_t = a_0 + \varepsilon_t$$

$$\Leftrightarrow \quad a(L)\, y_t = a_0 + \varepsilon_t$$

$$\Leftrightarrow \quad y_t = a(1)^{-1} a_0 + a(L)^{-1} \varepsilon_t$$

$$\Leftrightarrow \quad y_t = y^* + \psi(L)\, \varepsilon_t$$

p^{th} order Difference Equations
3. Using the Lag operator

Note:

$$a(L)\, y_t = a_0 + \varepsilon_t$$

$$\Leftrightarrow \quad y_t = y^* + a(L)^{-1}\, \varepsilon_t$$

$$\Leftrightarrow \quad a(L)\,(y_t - y^*) = \varepsilon_t$$

p^{th} order Difference Equations

4. IRF and stability conditions

$$y_t = y^* + (\varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \psi_3 \varepsilon_{t-3} + \cdots)$$

Impulse Response Function

What are the impacts of a **TRANSITORY** shock ε_t on the present and future values of y_t ? (multipliers ≡ IRF)

$$\partial y_{t+n} / \partial \varepsilon_t = \begin{cases} 1 & \text{for } n = 0 \\ \psi_n & \text{for } n = 1, 2, 3, \ldots \end{cases}$$

p^{th} order Difference Equations

4. IRF and stability conditions

How to factorize:

$$(1 - a_1 L - \cdots - a_p L^p)$$

$$= (1 - \lambda_1 L) \cdots (1 - \lambda_p L) \ ?$$

Find roots of characteristic polynomial:

i.e., solve: $\lambda^p - a_1 \lambda^{p-1} - \cdots - a_p = 0$

p^{th} order Difference Equations

4. IRF and stability conditions

Stability Condition:

All characteristic roots

$$\lambda_1 , \lambda_2 , \dots , \lambda_p$$

have to be inside the unit circle

p^{th} order Difference Equations
4. IRF and stability conditions

Another way to find $\lambda_1 , \lambda_2 , \dots ,\lambda_p$:

$1^{st} \rightarrow$ Find roots of Lag polynomial $\boldsymbol{a(L)}$

i.e., solve: $(1 - a_1 z - \cdots - a_p z^p) = 0$

and get roots $z_1 , z_2 , \dots ,z_p$.

$2^{nd} \rightarrow \lambda_1 = z_1^{-1} , \lambda_2 = z_2^{-1}, \dots ,\lambda_p = z_p^{-1}$

p^{th} order Difference Equations
4. IRF and stability conditions

Stability Condition:

All roots of Lag polynomial

$$z_1 , z_2 , \dots , z_p$$

have to be <u>outside</u> the unit circle

p^{th} order Difference Equations
4. IRF and stability conditions

Suppose <u>stability condition holds</u>, then:

Long-run (LR) impact on *y*:

- transitory shock in ε has no LR impact on *y*
- permanent shock in ε has a LR impact on *y*:

$$\psi(1) = a(1)^{-1}$$

p^{th} order Difference Equations
4. IRF and stability conditions

Additional resources

- Finding roots in Scientific Workplace/Wolfram|Alpha:
  - ➢sim_dif_eq.pdf
- Simulations in Gretl:
  - ➢sim_diff_eq_1/2/3.inp
- IRF in Gretl
  - ➢sim_diff_eq_2_irf.inp
- Simulations in Excel:
  - ➢excel diff eq order 1/2/3.xlsx