NOVA SCHOOL OF BUSINESS & ECONOMICS

MACROECONOMETRICS

Master in Economics

Difference Equations

Part I - Introduction

Examples of Time-Series Models

1. **Forecasting with an Unobserved Components Model**

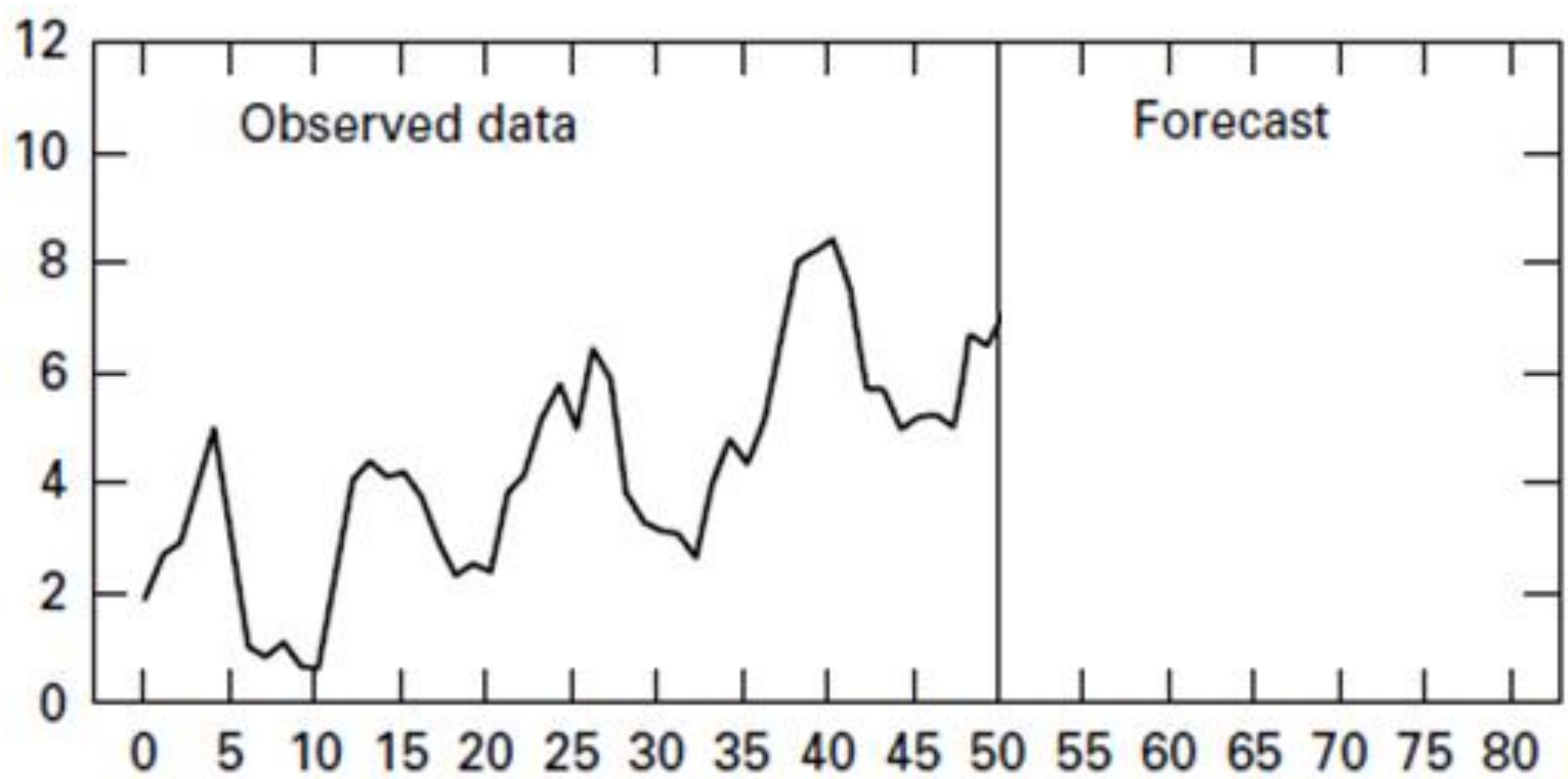
2. **Testing the Random Walk Hypothesis**

3. **Samuelson’s Classical Model**

4. **Cobweb Model**

1. Forecasting with an Unobserved Components Model

Forecasting Problem



1. Forecasting with an Unobserved Components Model

"Structural Time-Series" Model: Trend+Seasonal+Irregular

$$Y_t = T_t + S_t + I_t \quad \begin{cases} \text{Trend: } T_t = 1 + 0.1t \\ \text{Seasonal: } S_t = 1.6\sin(t\pi/6) \\ \text{Irregular: } I_t = 0.7\, I_{t-1} + \varepsilon_t \end{cases}$$

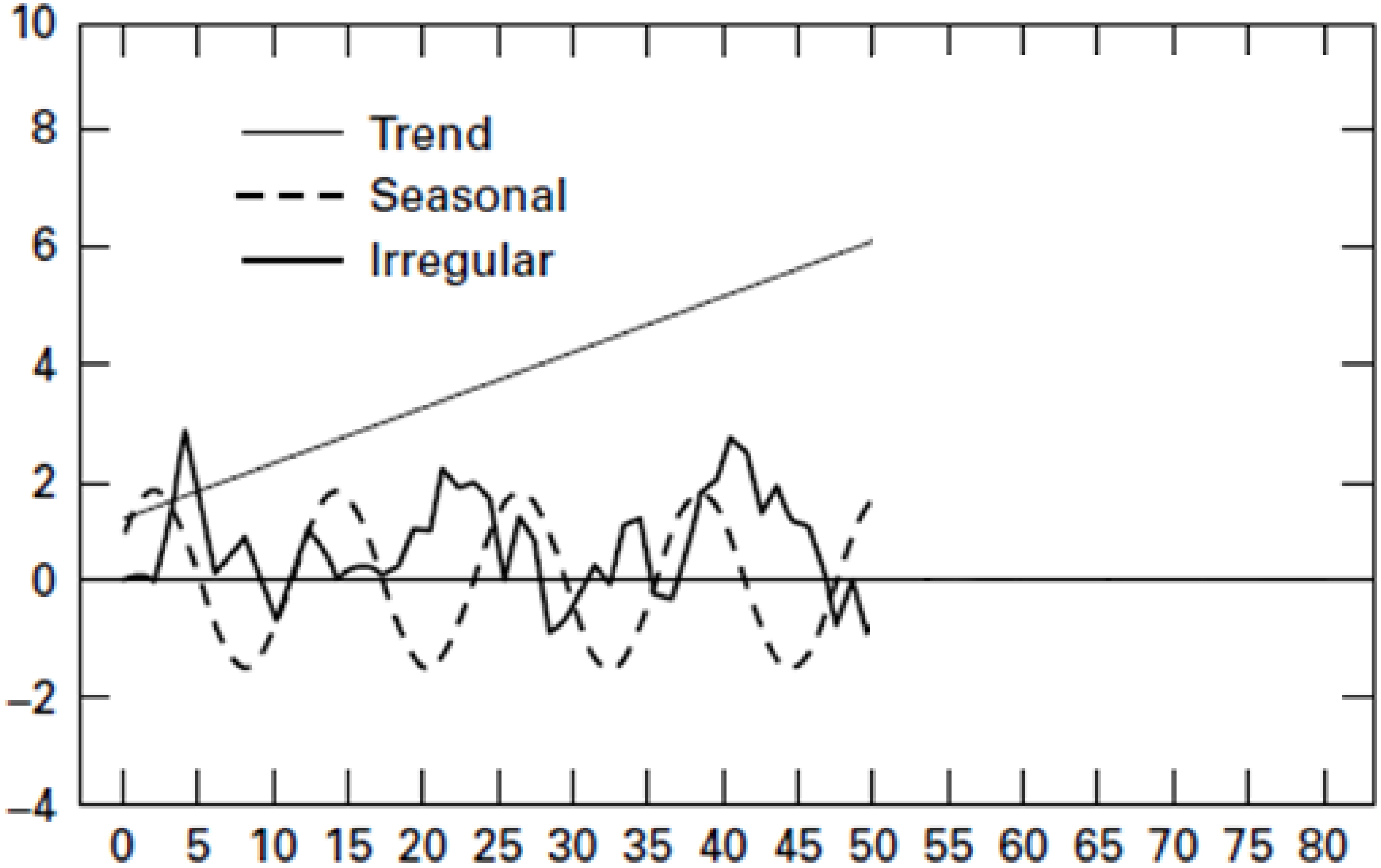
Notes:

- Trend is deterministic with slope 0.1
- Sine function generates a full cycle from 0 to 2π (degrees measured in radians)

  $\Rightarrow$ Seasonality cycles have a periodicity equal to 12: $t\,\pi / 6 = 2\,\pi \Leftrightarrow t = 12$
- Irregular has an autocorrelation of 0.7
- ε_t are uncorrelated random shocks with zero mean

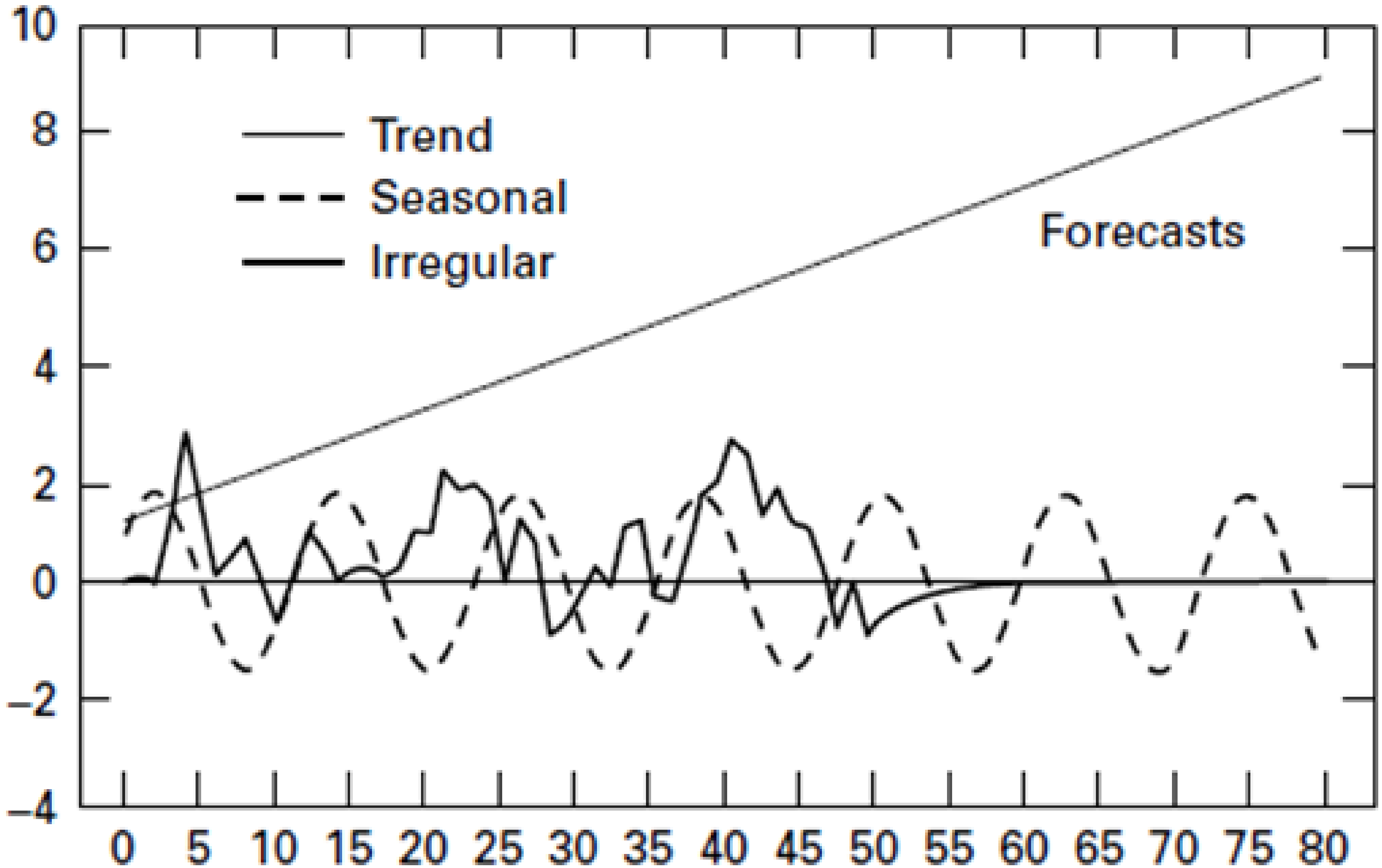
1. Forecasting with an Unobserved Components Model

Estimate the Unobserved Components



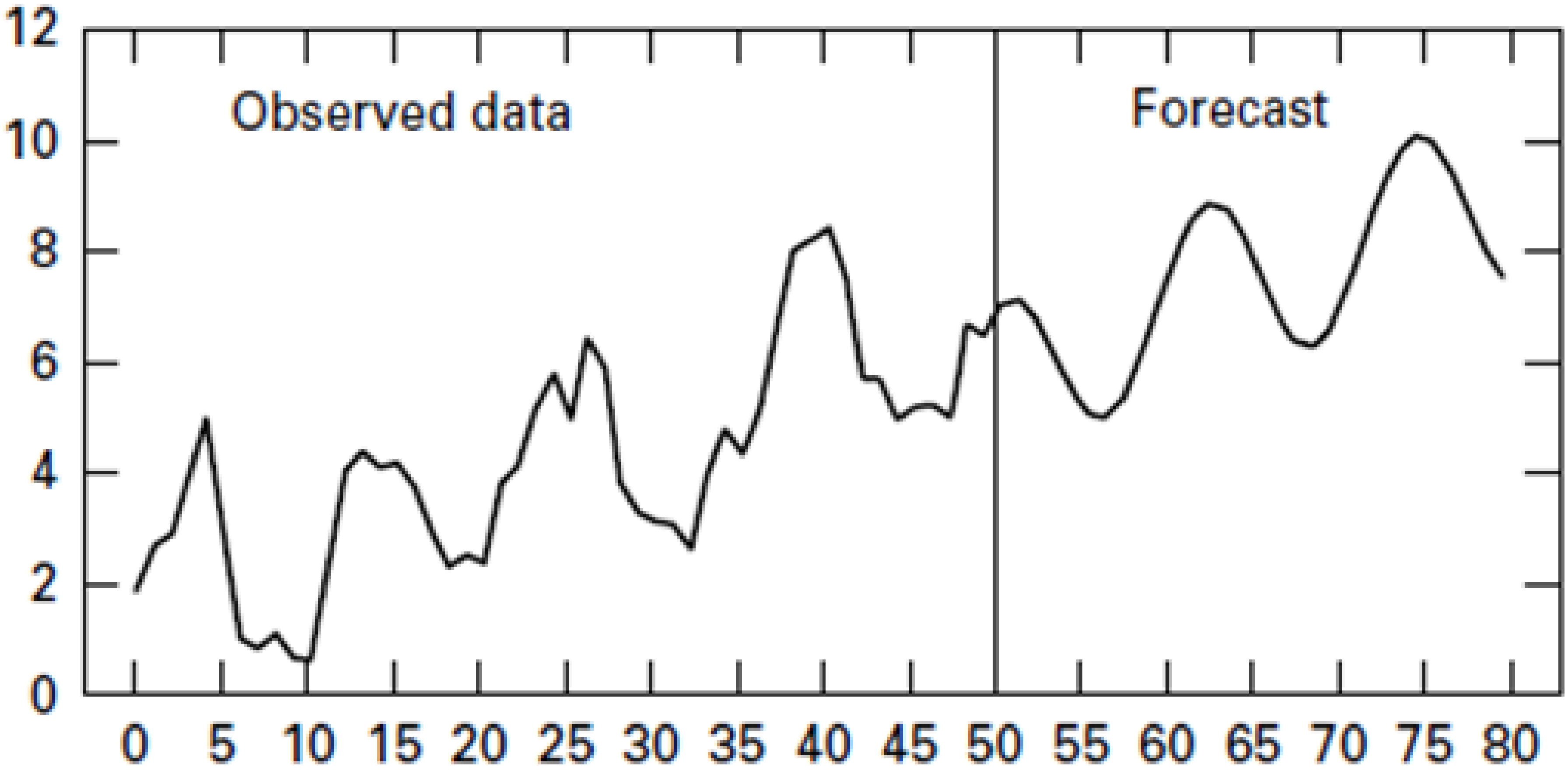
1. Forecasting with an Unobserved Components Model

Forecast the Components

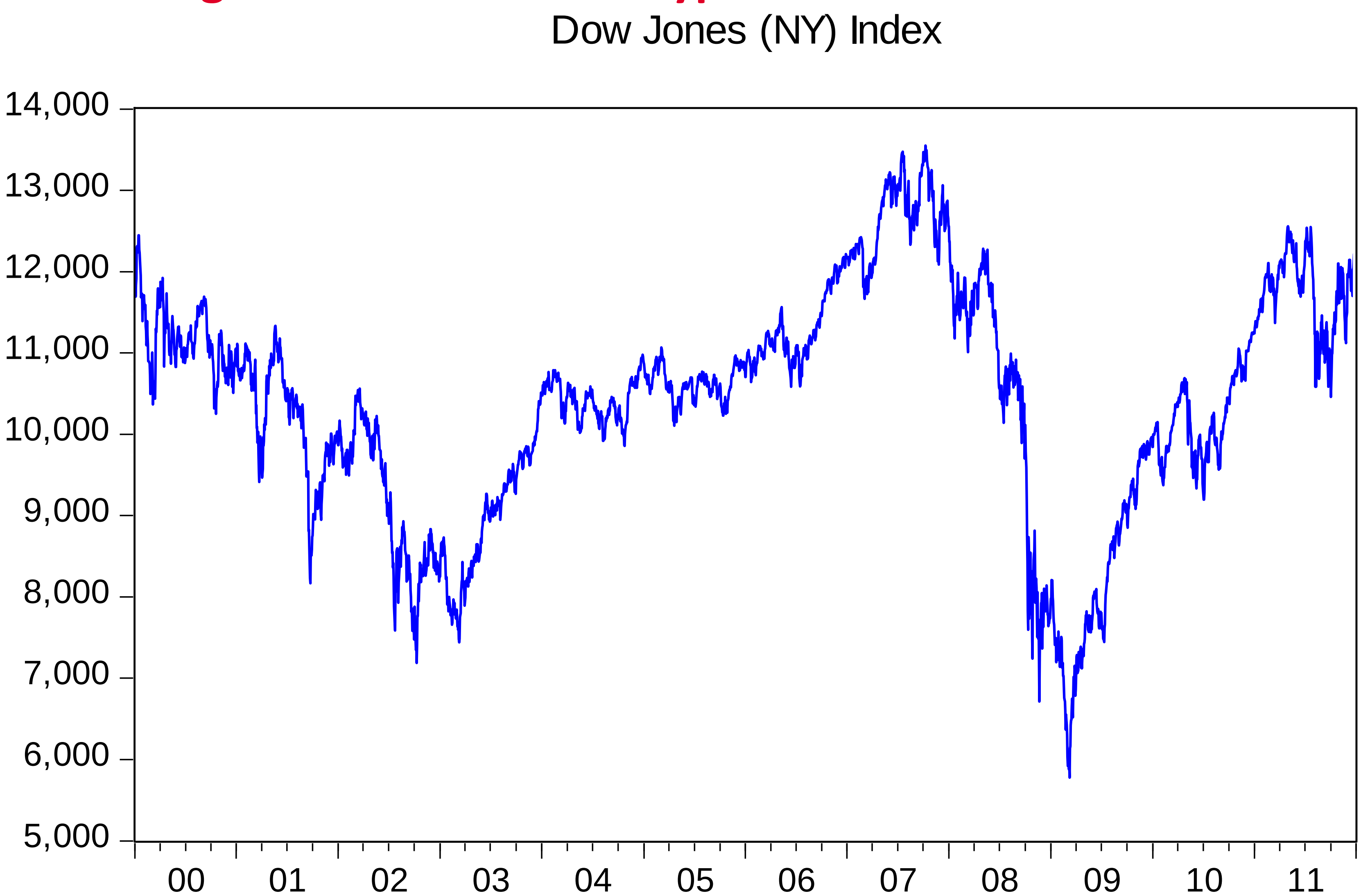


1. Forecasting with an Unobserved Components Model

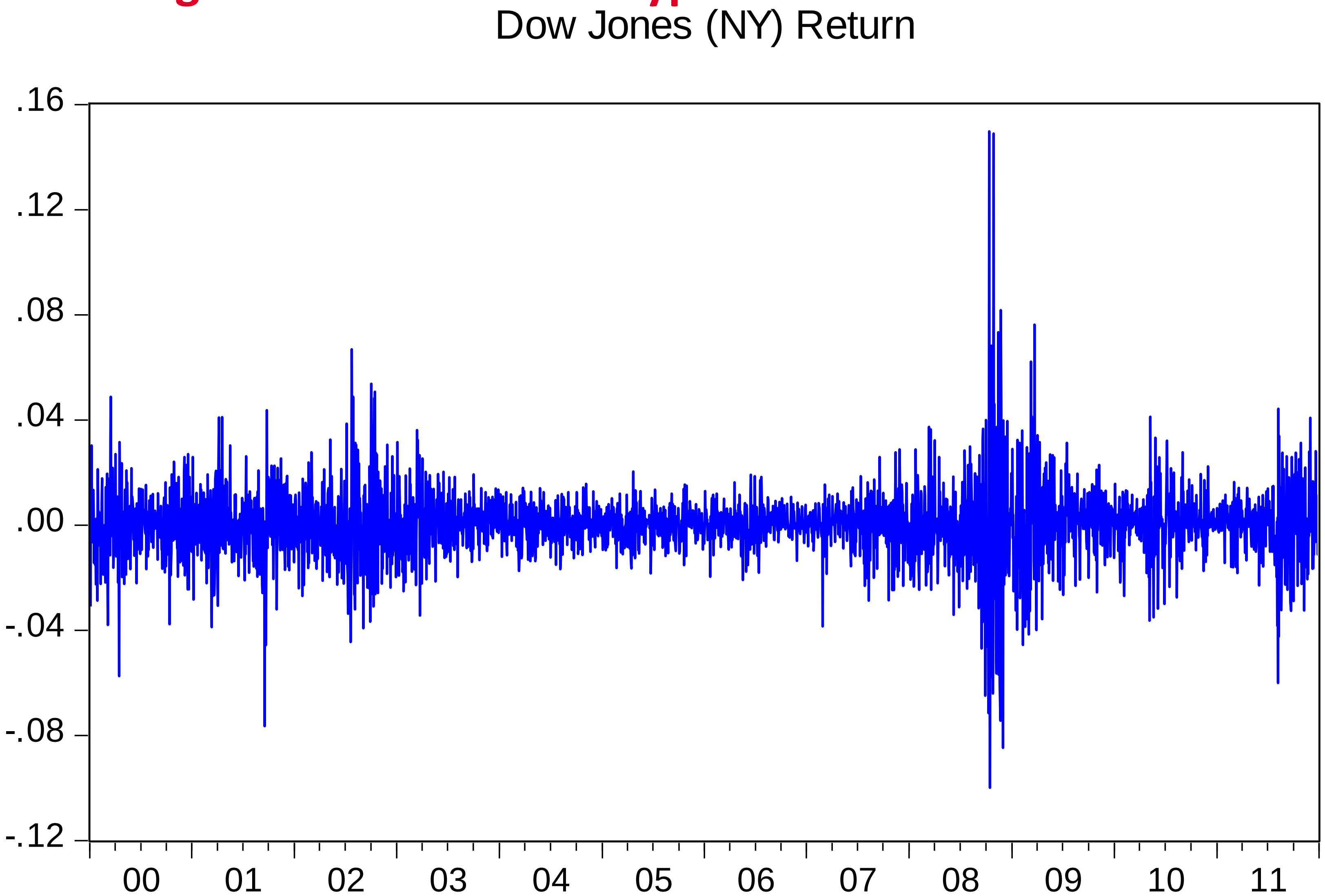
Forecast the Original Series



2. Testing the Random Walk Hypothesis



2. Testing the Random Walk Hypothesis



2. Testing the Random Walk Hypothesis

Random Walk: $$y_{t+1} = y_t + \varepsilon_{t+1}$$

or

$$\Delta y_{t+1} = \varepsilon_{t+1}$$

Test in a regression $$\Delta y_{t+1} = \alpha_0 + \alpha_1 y_t + \varepsilon_{t+1}$$

that: H_0: $\alpha_0 = \alpha_1 = 0$

3. Samuelson's Classical Model

Stochastic version of Samuelson' classic model:

$$\begin{cases} y_t = c_t + i_t & \\ c_t = \alpha y_{t-1} + \varepsilon_{ct} & 0 < \alpha < 1 \\ i_t = \beta(c_t - c_{t-1}) + \varepsilon_{it} & \beta > 0 \end{cases}$$

Endogenous variables: y_t , c_t , i_t

Predetermined variables: $y_{t\text{-}1}$, $c_{t\text{-}1}$

Stochastic disturbances: ε_{ct} , ε_{it}

3. Samuelson's Classical Model

Stochastic version of Samuelson' classic model:

$$\begin{cases} y_t = c_t + i_t \\ c_t = \alpha y_{t-1} + \varepsilon_{ct} & 0 < \alpha < 1 \\ i_t = \beta(c_t - c_{t-1}) + \varepsilon_{it} & \beta > 0 \end{cases}$$

Structural equation for investment

A reduced-form equation for investment

$$i_t = \beta[\alpha y_{t-1} + \varepsilon_{ct} - c_{t-1}] + \varepsilon_{it}$$

$$= \alpha\beta y_{t-1} - \beta c_{t-1} + \beta\varepsilon_{ct} + \varepsilon_{it}$$

3. Samuelson's Classical Model

Solving for c_t and i_t and substituting in y_t we get:

$$
\begin{aligned}
y_t &= \alpha y_{t-1} + \varepsilon_{ct} + \alpha\beta(y_{t-1} - y_{t-2}) + \beta(\varepsilon_{ct} - \varepsilon_{ct-1}) + \varepsilon_{it} \\
&= \alpha(1 + \beta)y_{t-1} - \alpha\beta y_{t-2} + (1 + \beta)\varepsilon_{ct} + \varepsilon_{it} - \beta\varepsilon_{ct-1}
\end{aligned}
$$

A univariate reduced-form equation for y_t

4. Cobweb Model

$$\begin{cases} d_t = a - \gamma p_t & \gamma > 0 \\ s_t = b + \beta p_t^* + \varepsilon_t & \beta > 0 \\ s_t = d_t \end{cases}$$

d_t = demand for wheat in period t

s_t = supply of wheat in t

p_t = market price of wheat in t

p_t^* = price that farmers expect to prevail at t

ε_t = a zero mean stochastic supply shock

4. Cobweb Model

Today is t-1 with:

Price = p_{t-1}

Quantity = s_{t-1}

What about future ?

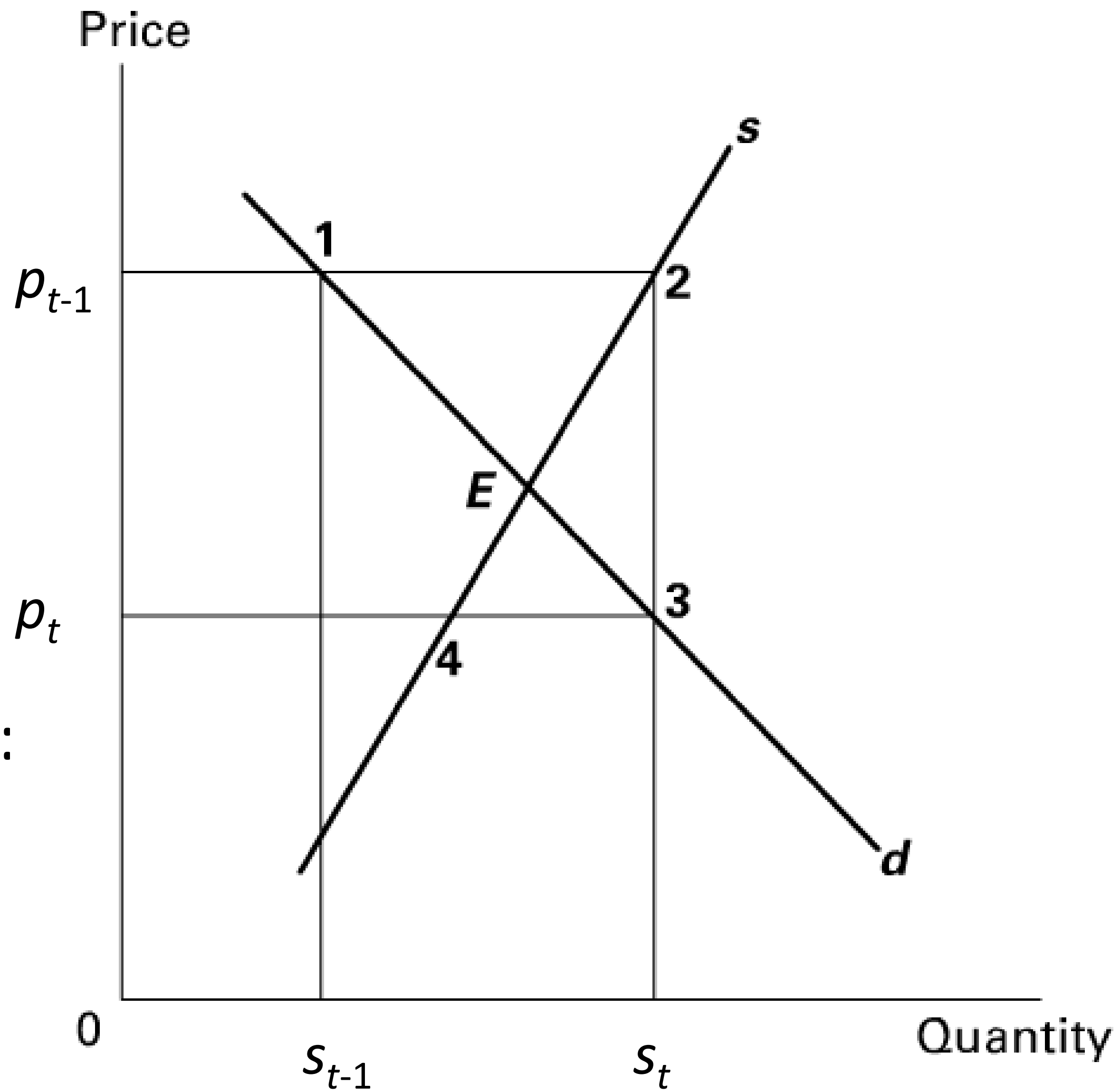
<u>Assumptions:</u>

1. Naive expectations:

$$p^*_t = p_{t-1}$$

2. No supply shocks:

$$\varepsilon_t = \varepsilon_{t+1} = \cdots = 0$$



4. Cobweb Model

If

Supply Curve steeper than Demand Curve:

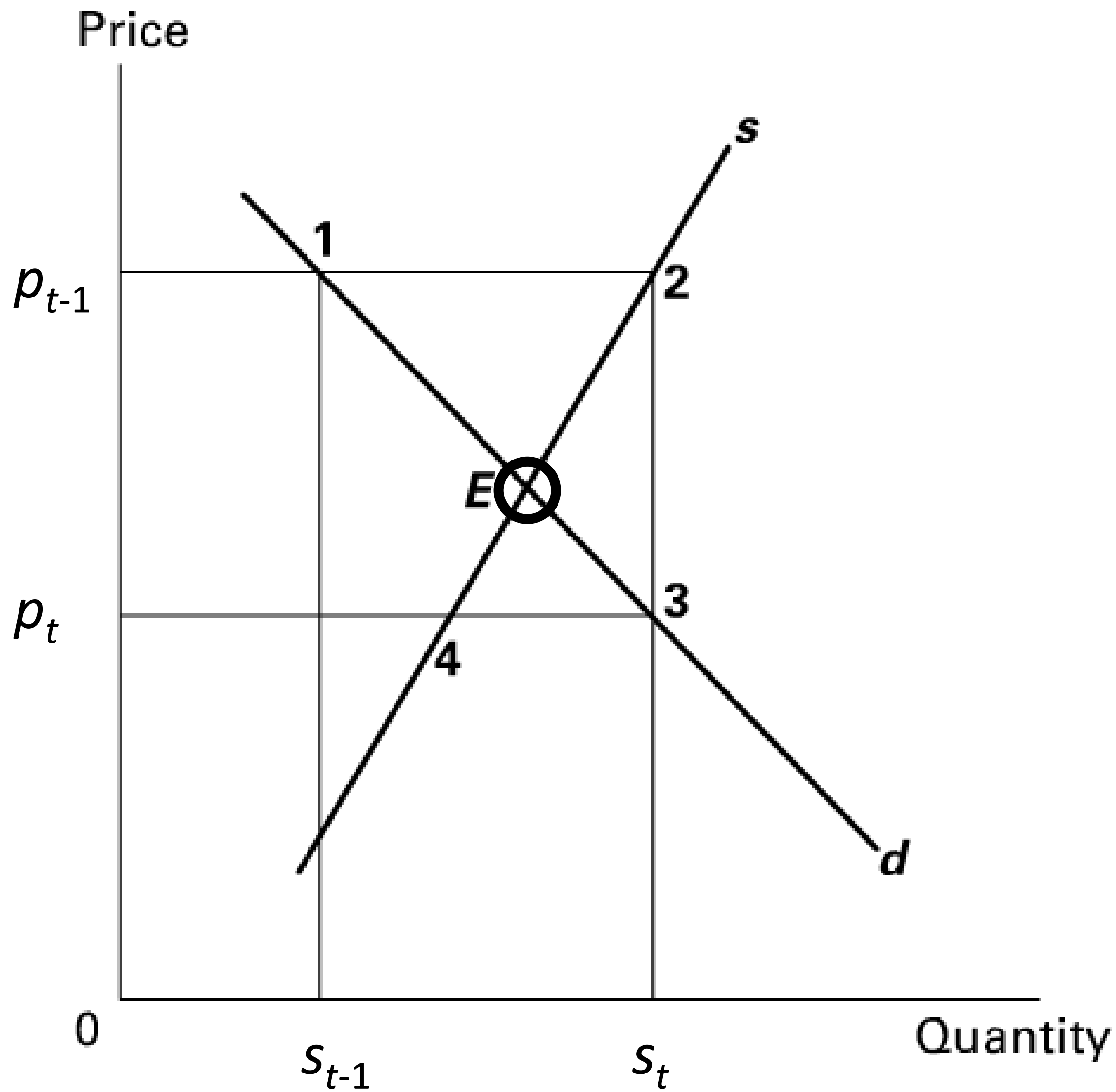
$$1/\beta > 1/\gamma$$

$$\Leftrightarrow$$

$$\beta/\gamma < 1$$

then

System converges to equilibrium point E



4. Cobweb Model

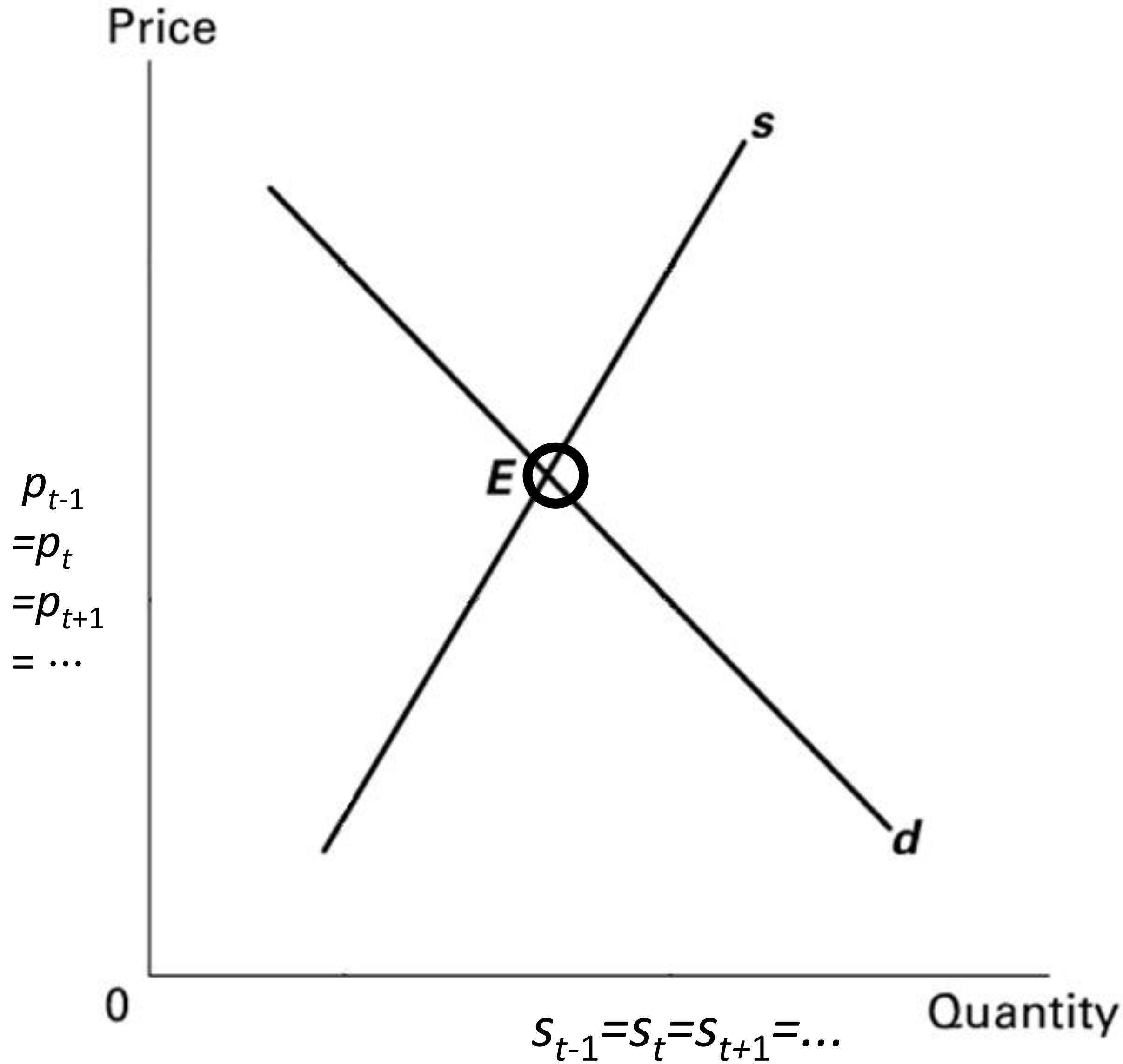
Suppose now in period t-1 we were already at point E

If at t, t+1, ⋯ there are no shocks:

$$\varepsilon_t = \varepsilon_{t+1} = \cdots = 0$$

then

$$p_{t-1} = p_t = p_{t+1} = \cdots$$



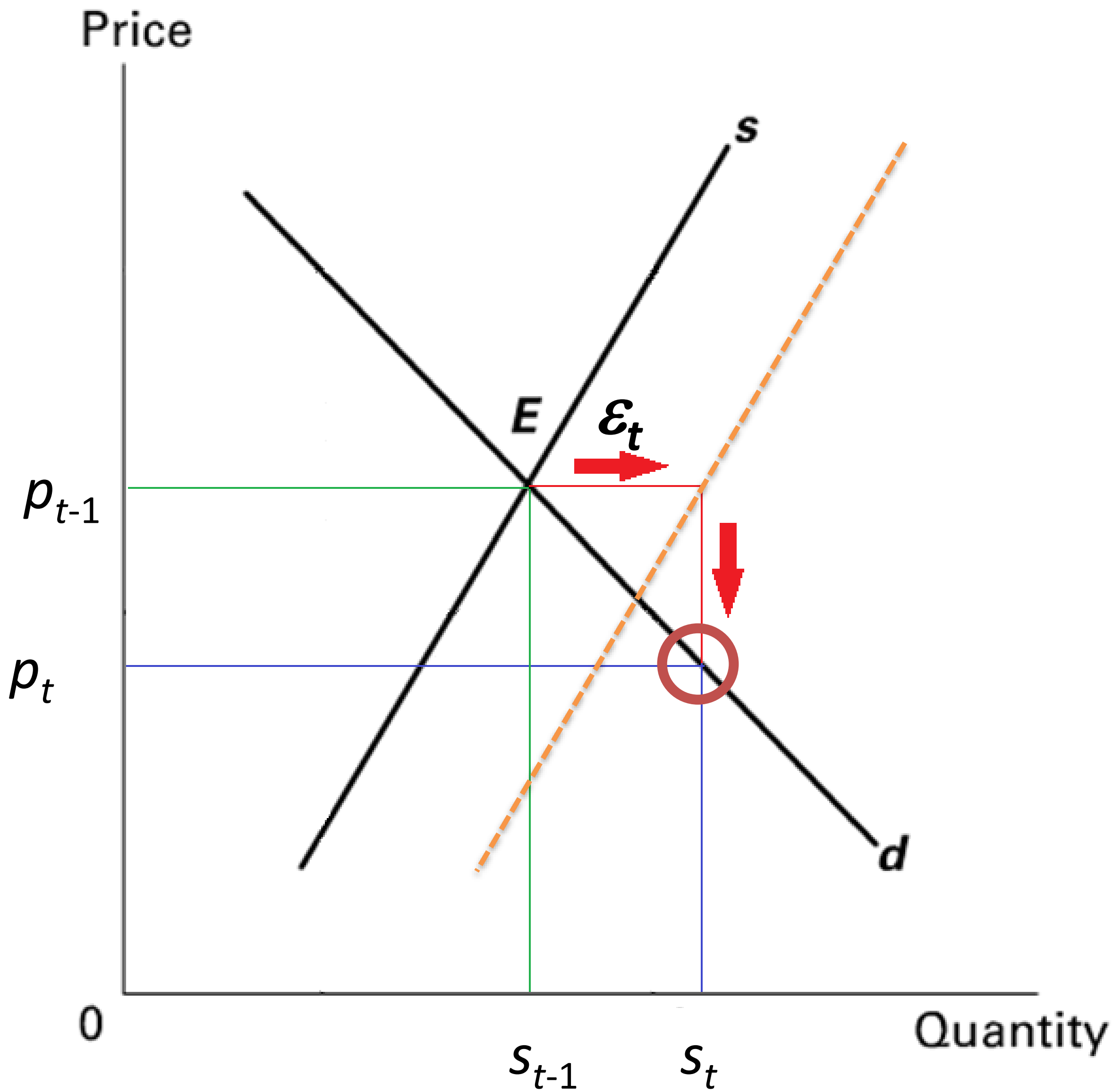
4. Cobweb Model

Suppose at
t-1 we were at *E*

If at *t*
there is a shock:
$\varepsilon_t \neq 0$

then

p_t will be below the value that would occur without shock



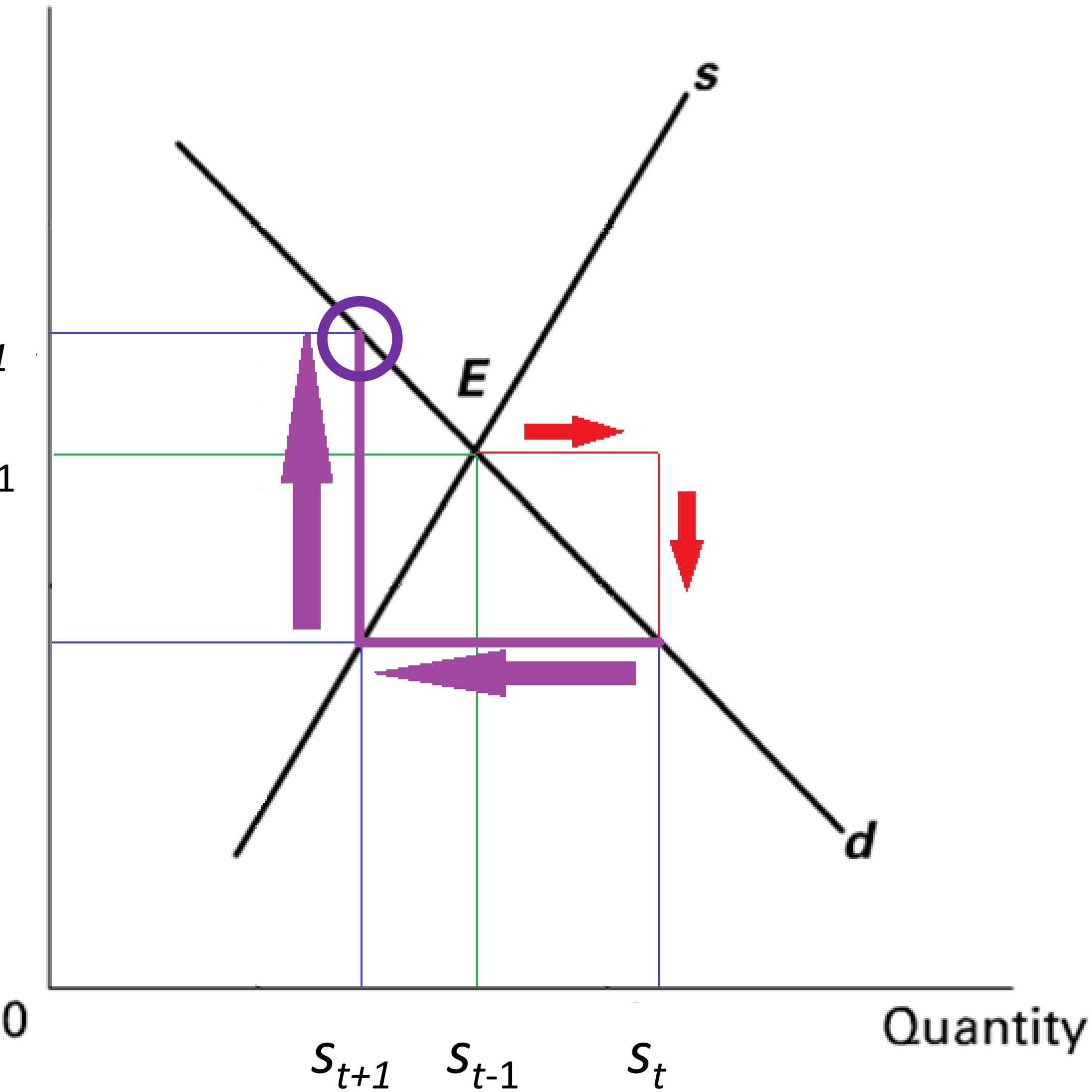
4. Cobweb Model

Suppose now $\varepsilon_t \neq 0$ but there are no further shocks

$\varepsilon_{t+1} = \varepsilon_{t+2} = \cdots = 0$

then

p_{t+1} will be above the value that would occur without the ε_t shock



4. Cobweb Model

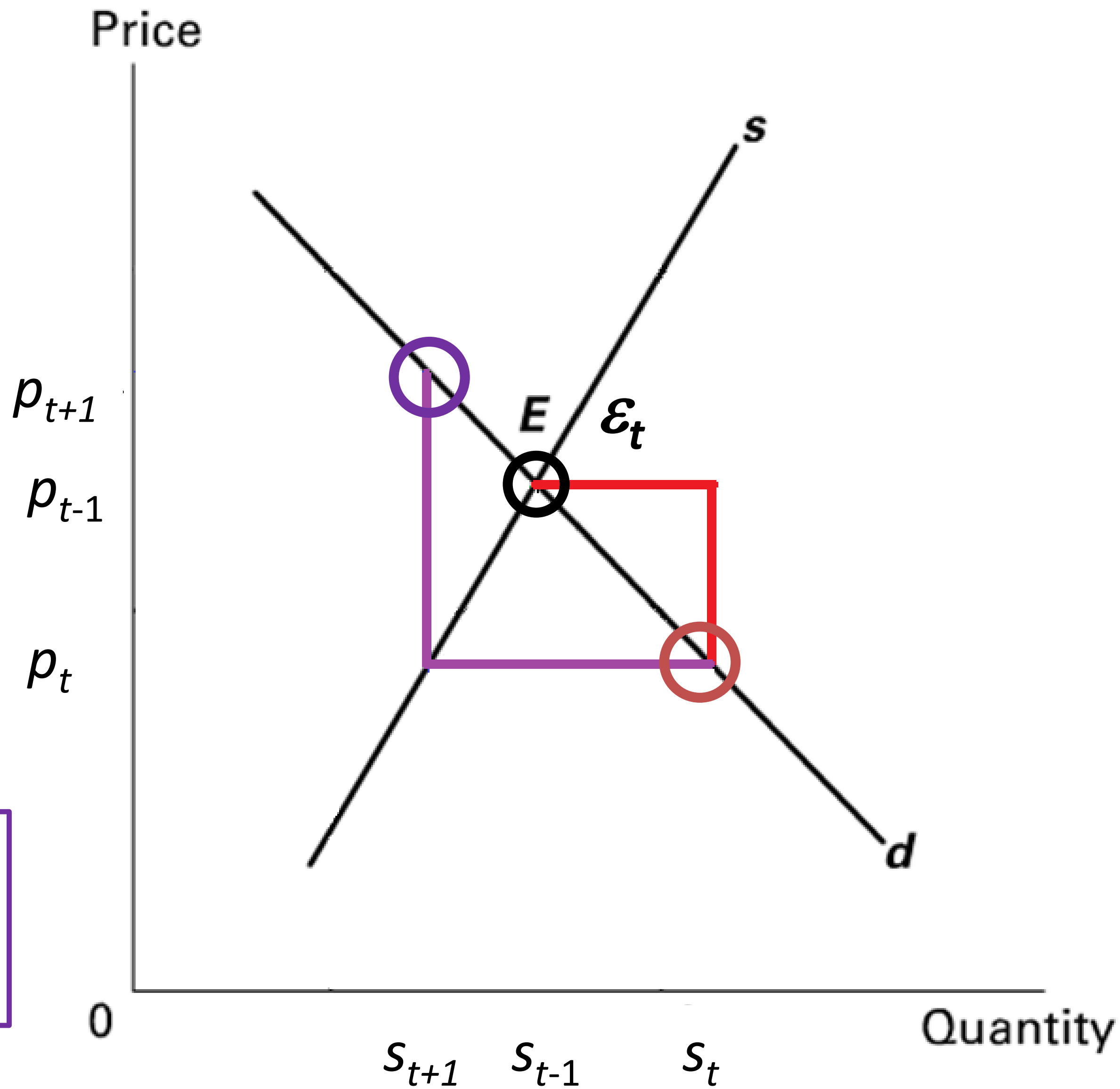
Analytically - Impact of the ε_t shock on prices is given by:

Impact multiplier:

$$\frac{\partial p_t}{\partial \varepsilon_t} = -1/\gamma$$

1-period multiplier:

$$\frac{\partial p_{t+1}}{\partial \varepsilon_t} = -(1/\gamma)(-\beta/\gamma)$$



4. Cobweb Model

2-period multiplier:

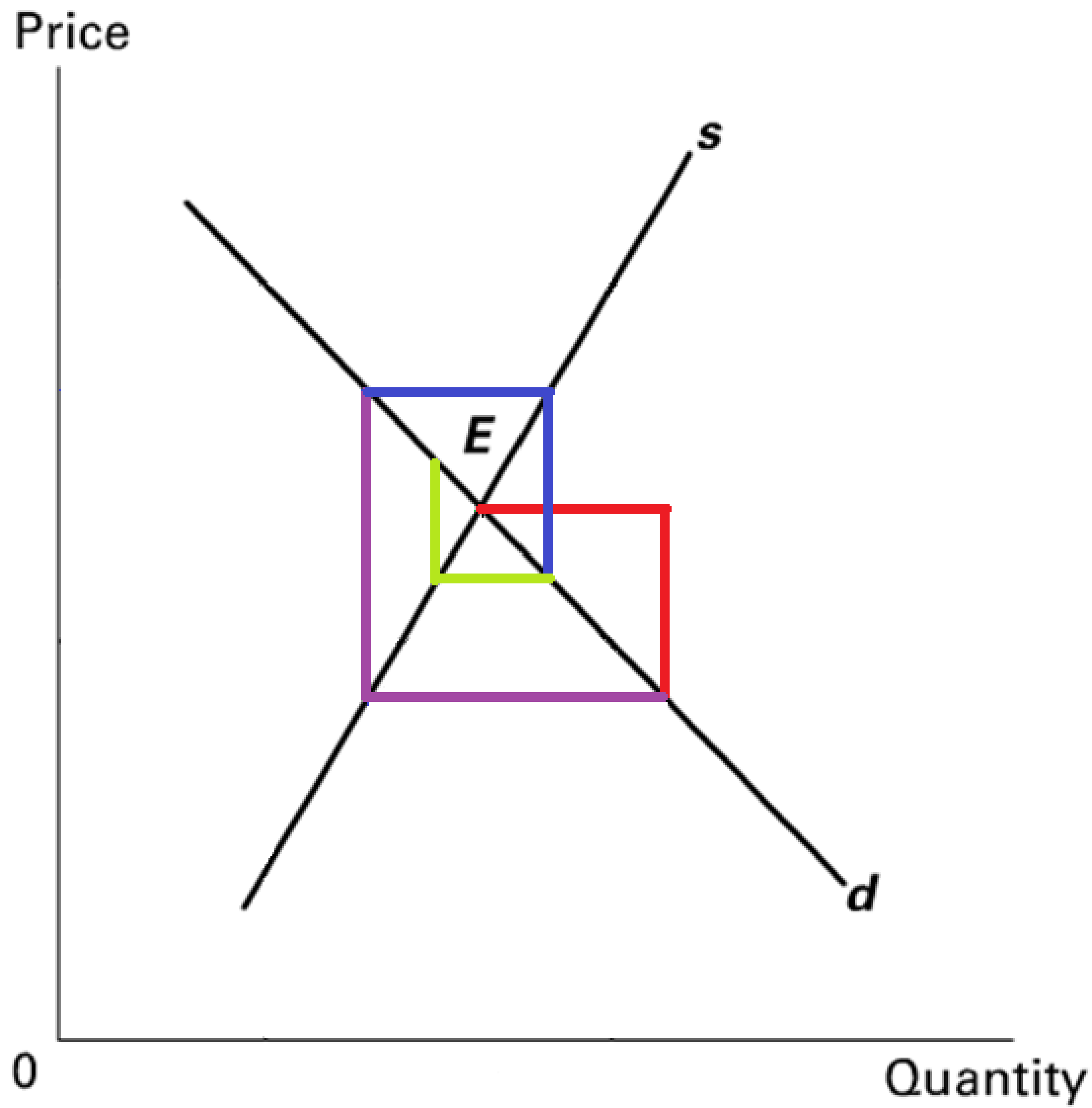
$$\frac{\partial p_{t+2}}{\partial \varepsilon_t} = -(1/\gamma)(-\beta/\gamma)^2$$

n-period multiplier:

$$\frac{\partial p_{t+n}}{\partial \varepsilon_t} = -(1/\gamma)(-\beta/\gamma)^n$$

Long-run multiplier:

$$\frac{\partial p_t}{\partial \varepsilon_t} = 0$$



4. Cobweb Model

Let's solve the model analytically:

$$\begin{cases} d_t = a - \gamma p_t & \gamma > 0 \\ s_t = b + \beta p_t^* + \varepsilon_t & \beta > 0 \\ s_t = d_t \\ p_t^* = p_{t-1} \end{cases}$$

$$b + \beta p_{t-1} + \varepsilon_t = a - \gamma p_t$$

4. Cobweb Model

Let's solve the model analytically:

$$b + \beta p_{t-1} + \varepsilon_t = a - \gamma p_t$$

$$p_t = (-\beta/\gamma)p_{t-1} + (a - b)/\gamma - \varepsilon_t/\gamma$$

This is a 1st order difference equation:

- Stability condition: $|-\beta/\gamma| < 1 \Leftrightarrow \beta/\gamma < 1$
- Long run equilibrium price (E) without shocks:

$$p = (a - b)/(\gamma + \beta)$$

4. Cobweb Model

Let’s solve the model analytically:

$$p_t = (-\beta/\gamma)p_{t-1} + (a - b)/\gamma - \varepsilon_t/\gamma$$

Start at time t = 0 assuming that price = p_0 and iterate forward:

$$p_t = \frac{a-b}{\gamma+\beta} - \frac{1}{\gamma}\sum_{i=0}^{t-1}(-\beta/\gamma)^i\varepsilon_{t-i} + \left(-\frac{\beta}{\gamma}\right)^t\left[p_0 - \frac{a-b}{\gamma+\beta}\right]$$