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Midterm

- **Date:** April 15th, 2023
- **Duration:** 2 hours
- **Instructions: 1:** The midterm has **six questions. 2:** Write your number and **absolutely nothing else** on this test paper, and **hand it in at the end. 3:** Write your answers on the answer booklet, using the **front and back** of each sheet, **stating** the question you are answering, **never** answering **more than one question on the same sheet**, and **not unstapling** any sheets. **4:** If you want to use any sheet of the answer booklet as space for **drafts,** state it clearly on the **space for the question number. 5: Show all your work, and properly justify** all your answers. **6: No** written support or calculators are allowed. **7:** You cannot leave the room before one hour has elapsed. **8: Mobile phones** must be **off** for the duration of the midterm. **9: When time is up** and you are prompted to do it, **photograph your answers and hand in the answer booklet.** Break a leg (not literally)!

Number:

1. **(4 points)** Consider the sets $A =]-1,1] \times [-1,2]$ and $B = B_1(0,0)$.

a. **(1 pt)** Represent on the same cartesian plane the sets A and B.

b. **(0.75 pts)** The proposition "$x \in A$" is a necessary condition, sufficient condition, or neither necessary nor sufficient for the proposition "$x \in B$"? Justify.

c. **(1 pt)** Consider the set $C = \{(x,y) \in \mathbb{R}^2 : 2^{y+1} < 4\}$. Represent, as a cartesian product of two intervals, the set $A \setminus C$.

d. **(1.25 pts)** Rewrite, without using the negation symbol (~), the proposition:

"$\sim \left(\forall (x,y) \in A , \left(\|(x,y)\| = \sqrt{2}\right) \Rightarrow (x \neq 1 \vee |y| < 1)\right)$"

For which point(s) is the proposition you obtained true?

2. **(3 points)**

a. **(1 pt)** Study, regarding convergence, the series $\sum_{n\geq 1} \frac{3^{n+1}}{2^n}$ and ~~$\sum_{n\geq 1} \left(2 + \frac{2}{n}\right)^{\frac{2}{n}}$~~.

b. **(1 pt)** For which values of x is the series $\sum_{n\geq 1} (4x-1)^n$ convergent? Find the sum of the series for $x = \frac{1}{3}$.

~~**c.**~~ **(1 pt)** Let (a_n) and (b_n) be sequences such that $a_n = \frac{n+1}{n^2}$ and $b_n < b_{n+1} < 5$. Consider the sequence $w_n = a_n \times b_n$. Can we ensure the existence of $\lim w_n$? If we can, compute it. If we cannot, explain why not.

3. **(4 points)** Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x,y) = \left(\log_2[x(y-1)], \frac{1}{e^{xy}-1}, \sqrt{2-x^2}\right)$.

a. **(1 pt)** Describe D, the domain of f, and represent it geometrically.

b. **(1.25 pts)** Describe the interior, exterior, derived set, and boundary of D.

c. **(0.75 pts)** Is D open? Is it compact? Is it connected?

d. **(1 pt)** Find the expression of the level curve with level zero of f_1, and represent it geometrically.

4. **(2 points)** Consider the functions $f: D_f \subset \mathbb{R}^2 \to \mathbb{R}^2$ and $g: D_g \subset \mathbb{R}^2 \to \mathbb{R}$, defined respectively by $f(x,y) = (\ln y, e^x)$ and $g(u,v) = \sqrt[3]{\frac{v^2}{e^{u-v}}}$.

a. **(1.25 pts)** Characterize, if possible, the functions $(f \circ g)$ and $(g \circ f)$, stating its domain, codomain, and general expression.

b. **(0.75 pts)** Is the function (or functions) you got in the previous part surjective (onto)?

5. **(3 points)** Consider the function $h: D_h \subset \mathbb{R}^3 \to CD_h = \mathbb{R}^3$, defined by the expression $h(x,y,z) = \left(\frac{1}{\ln x}, -\frac{1}{\sqrt{y}}, \frac{1}{z^2}\right)$.

a. **(1 pt)** Find D_h, and show that h is not invertible on D_h.

b. **(2 pts)** Restrict, if necessary, D_h and CD_h such that h becomes invertible and, in this case, characterize h^{-1}, the inverse function of h.

6. **(4 points)** Consider the function $g: D_g \subset \mathbb{R}^2 \to \mathbb{R}$, defined by $g(x,y) = \frac{x^3y^2+x^2y^3}{x^2+y^4}$.

a. **(1 pt)** Show that all iterated and directional limits of g at $(0,0)$ are equal to 0.

b. **(0.5 pts)** Considering only the results from part a), what can you conclude about the existence of limit of g at $(0,0)$?

c. **(1.5 pts)** Show, using the definition of limit of a function at a given point, that $\lim_{(x,y)\to(0,0)} g(x,y)$ exists.

d. **(1 pts)** Without further calculations, state the value of the following limit, explaining intuitively what it represents:

$$\lim_{x\to 0}\left(\frac{x^3(e^x-1)^2 + x^2(e^x-1)^3}{x^2+(e^x-1)^4}\right)$$

Solution Topics

1.

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- Represent sets $A = \{(x,y) \in \mathbb{R}^2 : -1 < x \leq 1 \wedge -1 \leq y \leq 2\}$, a rectangle, and $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, a circle.

a.

- State that $x \in B \Rightarrow x \in A$, that is, being an element of B implies being an element of A.
- State that $x \in A$ is a necessary condition for $x \in B$.

b.

- Solve $2^{y+1} < 4 \Leftrightarrow 2^y \times 2 < 4 \Leftrightarrow 2^y < 2^1 \Leftrightarrow y < 1$, therefore $C = \mathbb{R} \times]-\infty, 1[$.
- State that $A \backslash C =]-1,1] \times [1,2]$.

c.

- Considering that the negation of of $q \Rightarrow p$ is $q \wedge \sim p$, and that the negation of $p \wedge q$ is $\sim p \wedge \sim q$, we obtain: "$\exists (x,y) \in A : \left(\|(x,y)\| = \sqrt{2}\right) \wedge (x = 1 \wedge |y| \geq 1)$"
- State that the points for which the above proposition is true are points $(1,-1)$ and $(1,1)$.

2.

a.

- Compute $\lim \left(\frac{3^{n+1}}{2^n}\right) = \lim \left(\frac{3 \times 3^n}{2^n}\right) = 3 \lim \left(\frac{3}{2}\right)^n = +\infty$ OR state that $\sum_{n\geq 1} \frac{3^{n+1}}{2^n} = 3 \sum_{n\geq 1} \left(\frac{3}{2}\right)^n$ is a geometric series with common ratio $r = \frac{3}{2}$.
- Justify that since $\lim \left(\frac{3^{n+1}}{2^n}\right) \neq 0$, the series does not verify the necessary condition for convergence OR justify that the series is a geometric series with common ratio $|r| = \left|\frac{3}{2}\right| > 1$. Therefore the series diverges.
- Compute $\lim \left(2 + \frac{2}{n}\right)^{\frac{2}{n}} = 2^0 = 1$
- Justify that since $\lim_{n\to\infty} \left(2 + \frac{2}{n}\right)^{\frac{2}{n}} \neq 0$, the series does not verify the necessary condition for convergence, therefore the series diverges.

b.

- State that $\sum_{n\geq 1}(4x-1)^n$ is a geometric series with common ratio $r = (4x-1)$.
- State that the series converges if $|r| = |4x-1| < 1$.

- Solve the inequality: $-1 < 4x - 1 < 1$, therefore, $0 < x < \frac{1}{2}$. The geometric series converges for $x \in \left]0, \frac{1}{2}\right[$.
- State that if $x = \frac{1}{3}$, $\sum_{n\geq 1} \left(\frac{1}{3}\right)^n$ is a geometric series with common ratio $r = \frac{1}{3}$, hence $|r| < 1$. Hence, the series converges.
- Compute the sum of the series: $S_\infty = \frac{1}{3}\frac{1}{1-\frac{1}{3}} = \frac{1/3}{2/3} = \frac{1}{2}$.

c.

- Compute $\lim a_n = \lim \frac{n+1}{n^2} = \lim \frac{n\left(1+\frac{1}{n}\right)}{n^2} = \lim \frac{\left(1+\frac{1}{n}\right)}{n} = 0.$
- Justify that (b_n) is a bounded sequence, since $b_1 \leq b_n \leq 5$, OR justify that (b_n) is strictly increasing and bounded, therefore convergent. So there exists $b \in \mathbb{R} :\ \lim b_n = b$.
- Justify that $\lim w_n = 0$ since (w_n) is the product of a null sequence (a_n) and a bounded sequence (b_n) , OR justify that $\lim w_n = \lim a_n \times \lim b_n = 0 \times b = 0$.

3.

a.

- Correctly identifies the domain as $D = \{(x, y) \in \mathbb{R}^2\colon x(y-1) > 0 \wedge e^{xy} \neq 1 \wedge x^2 \leq 2\}$.
- Graphically represents the region $D = \{(x, y) \in \mathbb{R}^2\colon [(x > 0 \wedge y > 1) \vee (x < 0 \wedge y < 1)] \wedge xy \neq 0 \wedge -\sqrt{2} \leq x \leq \sqrt{2}\}$.

b.

- Computes the interior to be $\{(x, y) \in \mathbb{R}^2\colon x(y-1) > 0 \wedge xy \neq 0 \wedge x^2 < 2\}$.
- Computes the exterior to be $\{(x, y) \in \mathbb{R}^2\colon x(y-1) < 0 \vee x^2 > 2\}$.
- Computes the derived set to be $\{(x, y) \in \mathbb{R}^2\colon x(y-1) \geq 0 \wedge x^2 \leq 2\}$.
- Computes the boundary to be the union of $\{(x, y) \in \mathbb{R}^2\colon (x = -\sqrt{2} \wedge y \leq 1) \vee (x = \sqrt{2} \wedge y \geq 1) \vee x = 0\}$ with $\{(x, y) \in \mathbb{R}^2\colon (y = 0 \wedge -\sqrt{2} \leq x \leq 0) \vee (y = 1 \wedge -\sqrt{2} \leq x \leq \sqrt{2})\}$.

c.

- Justifies that D is not open because $D \neq \text{Int}(D)$.
- Justifies that D is not compact because it is not bounded, or because it is not closed.
- Justifies that D is disconnected by finding two sets A, B such that D = A ∪B and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, for example, $A = \{(x, y) \in \mathbb{R}^2\colon -\sqrt{2} \leq x < 0 \wedge y < 1 \wedge y \neq 0\}$, B=D\A .

d.

- Solves the equation $f_1(x, y) = 0$ and obtains $y = 1 + 1/x$.
- Graphically represents this hyperbola inside D.

4.

a.

- State that fog is not defined, the images of g are real numbers and the images of f are elements belonging to $\mathbb{R}^2$.
- State that $D_{gof} = \{(x,y) \in \mathbb{R}^2\colon y > 0 \ \wedge \ y \neq e^x\}$.
- Write the general expression of gof, given by $(gof)(x,y) = \sqrt[3]{\frac{e^{2x}}{y-e^x}}$, and state that $\mathbb{R}$ is its codomain.

b.

- State that gof is a surjective function if its range is equal to its codomain.

- Conclude that 0 belongs to $CC_{gof} = \mathbb{R}$ but not to CD_{gof} (we cannot find an object of gof whose image is 0 (e^{2x} is always different from 0).
- Conclude that gof is not a surjective function, $CD_{gof} \neq CC_{gof}$.

5.

a.

- Computes the domain to be $D_h = \{(x,\ y,\ z) \in \mathbb{R}^3\colon\ x > 0 \wedge x \neq 1 \wedge y > 0 \wedge z \neq 0\}$.
- Justifies that h is not invertible because it is not injective (or because it is not surjective). For example, $h(2,\ 1,\ 1) = h(2,\ 1,\ -1)$, which shows that h is not injective.

b.

- Restricts the domain correctly, for example, to $D = \{(x,\ y,\ z) \in \mathbb{R}^3\colon\ x > 0 \wedge x \neq 1 \wedge y > 0 \wedge z > 0\}$.
- Restricts the codomain correctly to $Y = \mathbb{R} \setminus \{0\} \times \mathbb{R}^- \times \mathbb{R}^+$ (in this example).
- Characterizes the inverse as the function $h^{-1}\colon Y \to D$ defined by $h^{-1}(u,\ v,\ w) = \left(e^{\frac{1}{u}},\ v^{-2},\ w^{-\frac{1}{2}}\right)$.

6.

a.

- Compute the iterated limits of g at $(0,0)$:

$$\lim_{x\to 0}\lim_{y\to 0}\left(\frac{x^3y^2+x^2y^3}{x^2+y^4}\right) = 0 \quad \text{and} \quad \lim_{y\to 0}\lim_{x\to 0}\left(\frac{x^3y^2+x^2y^3}{x^2+y^4}\right) = 0$$

- Compute the directinal limits of g at $(0,0)$:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}}\left(\frac{x^3y^2+x^2y^3}{x^2+y^4}\right) = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0)\\ x=0}}\left(\frac{x^3y^2+x^2y^3}{x^2+y^4}\right) = 0$$

b.

- State that the results obtained in a) are not sufficient to conclude that the limit of g at $(0,0)$ is 0. We can only say that, if the limit exists, it will be 0.
- State that we can have limits of g at $(0,0)$, along specific non linear paths, different from 0. If so, the conclusion is that there no limit of g at $(0,0)$.

c.

- Write $\forall\delta > 0, \exists\varepsilon > 0\colon 0 < ||(x,y) - (0,0)|| < \varepsilon \Rightarrow |g(x,y) - 0| < \delta$.
- Starting from $|g(x,y) - 0| < \delta$ find a relationship between ε and δ, valid for $\forall\delta > 0$. For example:

$$\left|\frac{x^3y^2+x^2y^3}{x^2+y^4}\right| = \frac{|x^3y^2+x^2y^3|}{x^2+y^4} \leq \frac{|x^3y^2+x^2y^3|}{x^2} \leq |xy^2 + y^3| \leq |x|y^2 + |y^3| \leq |x|y^2 + |y|y^2$$

$$\leq \left||x,y|\right|\left||x,y|\right|^2 + \left||x,y|\right|\left||x,y|\right|^2 = 2\left||x,y|\right|^3$$

If $2\left||x,y|\right|^3 < \delta$, we will have $\left||x,y|\right| < \sqrt[3]{\frac{\delta}{2}}$, therefore we can define $\varepsilon \leq \sqrt[3]{\frac{\delta}{2}}$.

- Conclude that $\forall\delta > 0, 0 < ||(x,y) - (0,0)|| < \sqrt[3]{\frac{\delta}{2}} \Rightarrow |g(x,y) - 0| < \delta$

d.

- Identify the given limit as the limit of g at $(0,0)$ along the specific non linear path $y = e^x - 1$.
- Conclude that, if 0 is the limit of g at $(0,0)$, then the only possible value for the given limit is 0.