Resit Exam

- **Date:** June 17th , 2023
- **Duration:** 2 hours and 30 minutes
- **Instructions: 1:** The exam has **five questions**. **2:** Write your number and **absolutely nothing else** on this test paper, and **hand it in at the end. 3:** Write your answers on the answer booklet, using the **front and back** of each sheet, **stating** the question you are answering, **never** answering **more than one question on the same sheet,** and **not unstapling** any sheets. **4:** If you want to use any sheet of the answer booklet as space for **drafts,** state it clearly on the **space for the question number. 5: Show all your work, and properly justify** all your answers. **6: No** written support or calculators are allowed. **7:** You cannot leave the room before one hour has elapsed. **8: Mobile phones** must be **off** for the duration of the exam. **9: When time is up** and you are prompted to do it, **photograph your answers and hand in the answer booklet.**

Break a leg (not literally)!

N°: *Please ignore exercises marked with a* ✖

1. **(4 points)** Consider $\boldsymbol{f}: \boldsymbol{D_f} \subset \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\boldsymbol{f(x,y)} = \left(\frac{x+y}{2-x^2-y^2}, \frac{2}{\sqrt{y-x^2}}\right)$.

a. **(1 pt)** Describe D_f and geometrically represent it.

b. **(0.75 pts)** Describe the derived set, the exterior and the boundary of D_f.

c. **(0.75 pts)** Is D_f connected? Is the complement of D_f path-connected?

d. **(0.75 pts)** Describe the level curves of f_2 with levels $\sqrt{2}$ and 2. Geometrically represent both.

e. **(0.75 pts)** Justify that f is not invertible.

2. **(3.5 points)** Let (u_n) be the sequence defined recursively, such that $u_1 = 4$ and $u_{n+1} = \frac{1}{3}u_n$. Consider the sets $A = \{u_n : n \in \mathbb{N}\}$ and $B =]2,3[$.

✖ **a.** **(0.75 pts)** Show that (u_n) is monotonic and bounded. Find its general term.

✖ **b.** **(0.5 pts)** Is (u_n) convergent? If so, find its limit.

c. **(0.75 pts)** Show that $\sum_{n\geq 1} u_n$ is convergent, and that its sum is equal to 6.

d. **(0.5 pts)** Let (S_n) be the sequence of the partial sums of (u_n). Find S_3 and $\lim S_n$.

e. **(1 pt)** Find the interior, the boundary, the closure, and the derived set of the set $(A \cup B)$.

3. **(6 points)** Consider the function $\boldsymbol{f}: \mathbb{R}^2 \to \mathbb{R}$, defined by:

$$f(x,y) = \begin{cases} \dfrac{y^3}{x^2+y^2} & \text{if} \quad y > x \\ x^2 + y & \text{if} \quad y \leq x \end{cases}$$

a. **(1.5 pts)** Show that f is continuous at $(0,0)$, but is not continuous at $(1,1)$.

b. **(1 pt)** Determine whether f is continuous at points (a,a), where $a \notin \{0,1\}$.

c. **(1 pt)** Show that $\nabla f(0,0) = (0,1)$.

d. **(1.5 pts)** Determine whether f is differentiable at $(0,0)$ and at $(1,1)$.

e. **(1 pt)** Compute $f'_{(1,1)}(0,0)$.

4. **(4.5 points)** Consider the functions $\boldsymbol{f}: \boldsymbol{D_f} \subset \mathbb{R}^3 \to \mathbb{R}^2$ and $\boldsymbol{g}: \boldsymbol{D_g} \subset \mathbb{R} \to \mathbb{R}^3$, defined respectively by $\boldsymbol{f(x,y,z)} = \left(\boldsymbol{\frac{1}{yz}}, \boldsymbol{\frac{1}{e^z - e}}\right)$ and $\boldsymbol{g(a)} = \left(\boldsymbol{\frac{1}{a-3}}, \boldsymbol{2a}, \boldsymbol{\ln(a-2)}\right)$.

a. **(1 pt)** Compute $\lim\limits_{a\to 3}[g_1(a) \times g_3(a)]$.

b. **(1.5 pts)** Let $h : \mathbb{R}\backslash\{3\} \subset \mathbb{R} \to \mathbb{R}$, be the function with expression $h(a) = |g_1(a)|$.

i. **(0.75 pts)** Show that $\forall a \in D_h$, $h'(a) \neq 0$.

ii. **(0.75 pts)** Does the result in **i)** contradict Rolle's theorem, when used referring to h on the interval $]2,4[$? Justify.

c. **(2 pts)** Characterize the function $(f \circ g)$.

5. **(2 points)** Consider $\boldsymbol{f}: \mathbb{R}^3 \to \mathbb{R}$, defined by $\boldsymbol{f(x,y,z) = x^m + yz}$, where $\boldsymbol{m} \in \mathbb{N}$.

Let $\boldsymbol{A} = \{(\boldsymbol{a,b,c}) \in \mathbb{R}^3 : \|(\boldsymbol{a,b,c})\| = \boldsymbol{1}\}$. It is known that $\boldsymbol{f}'_{\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)}(\boldsymbol{1,1,1}) = \boldsymbol{\sqrt{3}}$.

a. **(1 pt)** Find the value of m.

Note: in case you could not solve the previous part, assume in what follows that $m = 1$.

b. **(1 pt)** Consider $g: A \subset \mathbb{R}^3 \to \mathbb{R}$, defined by $g(a,b,c) = f'_{(a,b,c)}(1,1,1)$. Justify that $\sqrt{3}$ is the global maximum of g.

Solution topics

1.

a.

- Show that $D_f = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \neq 2 \ \wedge \ y > x^2\}$.
- Graphically represent the domain.

b.

- Write $D'_f = \{(x,y) \in \mathbb{R}^2 \colon \ y \geq x^2\}$.
- Write $ext(D_f) = \{(x,y) \in \mathbb{R}^2 \colon \ y < x^2\}$.
- Write $front(D_f) = \{(x,y) \in \mathbb{R}^2 \colon \ y = x^2 \ \vee \ (x^2 + y^2 = 2 \ \wedge \ y \geq x^2)\}$

c.

- Conclude that D_f is not connected, since:
  $\exists D_{f_1}, D_{f_2} \colon \ D_{f_1} \cup D_{f_2} = D_f, \overline{D}_{f_1} \cap D_{f_2} = \emptyset, \overline{D}_{f_2} \cap D_{f_1} = \emptyset$.
- Give an example of D_{f_1} and D_{f_2} that respect the conditions above. A possible example is:
  $D_{f_1} = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 > 2 \ \wedge \ y > x^2\}$ and $D_{f_2} = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 2 \ \wedge \ y > x^2\}$
- Graphically represent the complement of D_f or, alternatively, describe it.
- Conclude that the complement of D_f is a path-connected set, since it is possible to connect any two elements of the set with a straight line or curve that only includes elements of D_f.

d.

- Solve $f_2(x,y) = \sqrt{2}$ and obtain $y = x^2 + 2$.
- Solve $f_2(x,y) = 2$ and obtain $y = x^2 + 1$.
- Represent both parabolas graphically inside the domain of f.

e.

- Justify that f is not surjective, since the range of f does not coincide with its codomain.
- Write that the codomain of f is $\mathbb{R}^2$ and show that there is at least one element of the codomain of f that cannot be generated by f, that is, does not belong to the range of f. One such example is $(1,0)$. Note that f_2 can never be 0.
- Conclude that if f is not surjective, then f is not invertible.

2.

a.

- Show that $u_{n+1} - u_n = \frac{1}{3}u_n - u_n = -\frac{2}{3}u_n < 0 \,, \forall n > 1$ (since $u_n > 0 \,, \ \forall n \in \mathbb{N}$).
- Compute $u_2 - u_1 = \frac{4}{3} - 4 = -\frac{8}{3} < 0$, and conclude that $u_{n+1} - u_n < 0 \,, \ \forall n \in \mathbb{N}$

- Conclude that u_n is strictly decreasing, hence monotonic.
- Show that $0 < u_n \leq 4$, that is, $|u_n| \leq 4$, and conclude that u_n is bounded.
- Show that since u_n is a geometric sequence, $u_n = u_1 \times r^{n-1} = 4 \times \left(\frac{1}{3}\right)^{n-1} = \frac{4}{3^{n-1}}$.

b.

- State that u_n is monotonic and bounded, hence, convergent.
- Compute $\lim u_n = \lim \frac{4}{3^{n-1}} = \lim 4 \times \lim \frac{1}{3^{n-1}} = 4 \times 0 = 0$.

OR

- Compute $\lim u_n = \lim \frac{4}{3^{n-1}} = \lim 4 \times \lim \frac{1}{3^{n-1}} = 4 \times 0 = 0$.
- State that since the limit is finite, the sequence is convergent.

c.

- State that $\sum_{n\geq1} u_n = \sum_{n\geq1} \frac{4}{3^{n-1}}$ is a geometric series with common ratio $r = \left(\frac{1}{3}\right)$.
- State that since $|r| = \left|\frac{1}{3}\right| < 1$, the geometric series is convergent, therefore its sum can be computed using the formula $\frac{u_1}{1-r}$.
- Compute the sum of the series as $\frac{4}{1-1/3} = \frac{4}{2/3} = \frac{12}{2} = 6$.

d.

- Compute $S_3 = u_1 + u_2 + u_3 = 4 + \frac{4}{3} + \frac{4}{9} = \frac{52}{9}$
- State that $\lim S_n$ is the sum of the series, which we are told to be equal to 6.

e.

- State that the interior of $(A \cup B)$ is $]2,3[$.
- State that the boundary of $(A \cup B)$ is $A \cup \{0, 2, 3\}$.
- State that the closure of $(A \cup B)$ is $A \cup [2,3] \cup \{0\}$.
- State that the derived set of $(A \cup B)$ is $[2,3] \cup \{0\}$.

3.

a.

- Computes the limit as $(x, y) \to (1, 1)$ through $y \leq x$ and gets the value 2. Computes the limit through the other branch obtaining $1/2$. Concludes that the function is not continuous at $(1, 1)$.
- Computes the limit as $(x, y) \to (0, 0)$ through $y \leq x$ and obtains 0.
- Shows that the limit as $(x, y) \to (0, 0)$ through $y > x$ is equal to zero by using the appropriate estimates. Concludes that the function is continuous at $(0, 0)$.

b.

- Computes the limit through both branches at concludes that the function is continuous at $(a, a), a \neq 0$, if and only if $a^2 + a = a/2$.
- Solves this equation and concludes that the function is continuous at $(a, a), a \neq 0$, if and only if $a = -1/2$.

c.

- Computes the partial derivative with respect to x using the definition.
- Understands that both lateral limits need to be checked and obtains zero in both cases.
- Concludes that $f'_x(0,0) = 0$.
- Does the same for the partial derivative with respect to y obtaining 1 through both branches.
- Concludes that $f'_y(0,0) = 1$.

d.

- Argues that the function is not differentiable at $(1,1)$, since it is not continuous at this point.
- Computes the remainder around $(0,0)$ to be $R(x,y) = f(x,y) - y$.
- Shows that the limit of $R(x,y)/||(x,y)||$ as $(x,y) \to (0,0)$ is not equal to zero. For example, this can be done by computing the limit through the trajectory $y = 2x, x > 0$. Concludes that the function is not differentiable at $(0,0)$.

e.

- Since the function is not differentiable at $(0,0)$, we cannot assume that the directional derivative will be equal to $\nabla f(0,0) \cdot (1,1)$. This way, the directional derivative must be computed using the definition.
- Understands that $f(t,t) = t^2 + t$, and uses this to compute the limit, obtaining $f'_{(1,1)}(0,0) = 1$.

4.

a.

- Uses Cauchy's rule to simplify the limit of $\ln(a-2)/(a-3)$ to the limit of $1/(a-2)$.
- Concludes that the limit is equal to 1. Alternatively, it is possible to compute the limit by comparing it to a special limit.

b.

i.

- When $a > 3$, $h'(a) = -(a-3)^{-2}$.
- When $a < 3$, $h'(a) = (a-3)^{-2}$.
- Concludes that $h'(a) \neq 0, \forall a \in D_h$.

ii.

- Justifies that h is not continuous in $[2,4]$, and therefore Rolle's theorem is not applicable.
- Alternatively, one can argue instead that h is not differentiable in $[2,4]$.

c.

- Computes the domain of the composition by the definition $D_{f\circ g} = \{a \in D_g: g(a) \in D_f\}$, obtaining $\{a \in \mathbb{R}: a > 2, a \neq 3, a \neq 2+e\}$.
- Computes the general expression of the composition and obtains $\left(\left(2aln(a-2)\right)^{-1}, (a-2-e)^{-1}\right)$.

5.

a.

- State that f is a function differentiable in its domain ($\mathbb{R}^2$) since it is a polynomial function, hence it is differentiable at $(1,1,1)$.
- State that since f is differentiable at $(1,1,1)$, then $f'_{\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)}(1,1,1) = \nabla f(1,1,1).\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)$.
- Write $\nabla f(1,1,1) = (mx^{m-1}, z, y)_{(1,1,1)} = (m,1,1)$.
- Write $(m,1,1).\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right) = \sqrt{3}$ and conclude that $m = 1$.

b.

- State that f is differentiable at $(1,1,1)$, therefore, among all the vectors with norm 1, the one generating the highest value for the directional derivative of f at $(1,1,1)$ is the normalized gradient vector of f at that point.
- State that the values of g are the values of the directional derivative of f at $(1,1,1)$ along normalized vectors (norm 1) (which are elements of A), then the global maximum of g is attained when (a,b,c) is equal to $\frac{\nabla f(1,1,1)}{||(1,1,1)||} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)$.
- Conclude that the global maximum of g is given by $g\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)$, that is, $\sqrt{3}$.