Regular Exam

- **Date:** May 19th , 2023
- **Duration:** 2 hours and 30 minutes
- **Instructions: 1:** The exam has **five questions**. **2:** Write your number and **absolutely nothing else** on this test paper, and **hand it in at the end. 3:** Write your answers on the answer booklet, using the **front and back** of each sheet, **stating** the question you are answering, **never** answering **more than one question on the same sheet**, and **not unstapling** any sheets. **4:** If you want to use any sheet of the answer booklet as space for **drafts**, state it clearly on the **space for the question number**. **5: Show all your work, and properly justify** all your answers. **6: No** written support or calculators are allowed. **7:** You cannot leave the room before one hour has elapsed. **8: Mobile phones** must be **off** for the duration of the exam. **9: When time is up** and you are prompted to do it, **photograph your answers and hand in the answer booklet**.

Break a leg (not literally)!

Nº:

1. **(2.5 points)** Consider the following sets of real numbers:

$$A = \{2 - \frac{2}{n} : n \in \mathbb{N}\} \qquad B =]0, k[\cap Q\ ,\ k \in \mathbb{R}^+ \qquad C = A \cup B$$

a. **(1.5 pts)** If $k = 1$:

i. **(0.75 pts)** Describe the interior, the boundary, and the derived set of C.

ii. **(0.75 pts)** Is C open? Is C compact? Is C connected?

b. **(1 pt)** Consider now k such that $C' = [0,2]$. Is $\sum_{n\geq 1} k^{-2n}$ a convergent series? If yes, find its sum. If not, explain why.

2. **(6 points)** Let $f: D_f \subset \mathbb{R}^2 \to R_f$, $g: \mathbb{R}^2 \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}^2$ be the functions defined by:

$$f(x,y) = (\ln(x+y), \ln(3x-y)) \qquad g(s,t) = e^{s+t} \qquad h(x) = (x, 2x)$$

a. **(1 pt)** Find D_f and represent it geometrically.

b. **(1 pt)** Find the expression of the level curve of f_2 with level 1, and of the level curve of f_2 which contains the point $(1,1)$. Represent both geometrically.

c. **(1.5 pts)** Is f invertible? If yes, characterize f^{-1}. If not, explain why.

d. **(1.5 pts)** Characterize, if possible, the functions $(g \circ f)$ and $(f \circ g)$.

e. **(1 pt)** Show that the graph of the function $(g \circ h)$ intersects the line with equation $y = -x$ on $]-1,1[$.

3. **(5 points)** Let $f: D_f \subset \mathbb{R}^2 \to \mathbb{R}$, be the function defined by:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & , \quad y < 0 \\ 0 & , \quad y \geq 0 \end{cases}$$

- **a.** **(1.5 pts)** Define the continuity domain of f.
- **b.** **(1.5 pts)** State and justify whether f is differentiable at $(0,0)$.
- **c.** **(1 pt)** Compute $f'_{(1,-1)}(2,0)$.
- **d.** **(1 pt)** State, and justify, the values of $k > 0$ for which the Weierstrass theorem ensures the existence of a global maximum of f in $A = B_1(0,0) \cup [1-k, 1+k]^2$.

4. **(3 points)** Let $f: D_f \subset \mathbb{R}^2 \to \mathbb{R}$ and $g: D_g \subset \mathbb{R}^2 \to \mathbb{R}$ be the functions defined by:

$$f(x,y) = \frac{e^{x+y^2}-1}{x+y} \qquad\qquad g(x,y) = \begin{cases} f(x,y) & , \ (x,y) \neq (0,0) \\ 0 & , \ (x,y) = (0,0) \end{cases}$$

- **a.** **(0.75 pts)** Show that the iterated limits of f at $(0,0)$ are different.
- **b.** **(1 pt)** Compute the directional limit of f at $(0,0)$ along the path $y = x$.
- **c.** **(0.75 pts)** Discuss whether g can be the result of a continuous extension of f to $(0,0)$.
- **d.** **(0.5 pts)** State, and justify, whether g is differentiable at $(0,0)$.

5. **(3.5 points)** Consider $f: \mathbb{R}^2 \to \mathbb{R}$, a class C^2 function, such that:

$$f'_x\,(1,0) = -f''_{yx}\,(1,0) = 1 \qquad\qquad f'_{(2,3)}\,(1,0) = 8$$

- **a.** **(0.75 pts)** Justify that f, f'_x and f'_y are differentiable in $\mathbb{R}^2$.
- **b.** **(1 pt)** Show that $\nabla_f(1,0) = (1,2)$.
- **c.** **(0.75 pts)** Compute $f''_{xy}(1,0)$.
- **d.** **(1 pt)** Let $(a,b) \in \mathbb{R}^2$. Discuss the truth value of the following proposition:

$$[(f'_{(a,b)}(1,0) = f''_{yx}(1,0)) \wedge \|(a,b)\| = 1] \Rightarrow (a,b) = (-1,0)$$

Solution topics

1.

a.

i.

- State that $int(C) = \emptyset$, $front(C) = [0,1] \cup A \cup \{2\}$ and $C' = [0,1] \cup \{2\}$.

ii.

- State that $int(C) \neq C$, so C is not open.
- State that $\bar{C} = [0,1] \cup A \cup \{2\} \neq C$, so C is not closed. If it is not closed, it is not a compact set (closed and bounded).
- Conclude that C é not connected. In fact, there exists C_1, C_2 such that $C_1 \cup C_2 = C$, $\overline{C_1} \cap C_2 = \emptyset$ and $\overline{C_2} \cap C_1 = \emptyset$. For example, $C_1 = C \cap \,]-\infty, 1]$ and $C_2 = C \cap \,]1, +\infty]$.

b.

- Conclude that $C' = [0,2] \Rightarrow k = 2$.
- For $k = 2$, write $\sum_{n\geq 1} 2^{-2n} = \sum_{n\geq 1} \left(\frac{1}{4}\right)^n$ and conclude that this is a geometric series with common ratio $r = \frac{1}{4}$.
- Conclude that, since $|r| = \left|\frac{1}{4}\right| < 1$, the geometric series is convergent.
- Compute the sum of the series, $S = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$.

2.

a.

- Show that $D_f = \{(x, y) \in \mathbb{R}^2 \colon -x < y < 3x\}$.
- Graphically represent the domain.

b.

- Solve $f_2(x, y) = 1$ and obtain $y = 3x - e$.
- Compute the level for the level curve that goes through $(1,\ 1)$ and obtain $k = \ln(2)$.
- Compute the corresponding level curve to be $y = 3x - 2$.
- Graphically represent the two lines inside the domain of f.

c.

- Justify that the function is bijective.
- Compute the expression of the inverse function and obtain $f^{-1}(u,\ v) = \left(\frac{e^u+e^v}{4},\ \frac{3e^u-e^v}{4}\right)$.

- State that the domain of the inverse is the range of the original function, and the range of the inverse is the domain of the original function, thus characterizing the inverse function (together with the expression above).

d.

- Justify that $f \circ g$ does not exist.
- Show that the domain of $g \circ f$ is D_f.
- Compute the expression of the composite function to be, $(g \circ f)(x,\ y)\ =\ (x+y)(3x-y)$.

e.

- Compute $(g \circ h)(x) = e^{3x}$
- Define a new function, $m(x) = e^{3x} + x$, and check that the corollary to Bolzano's theorem applies to this function on the interval $[-1, 1]$.

3.

a.

- Justify that the function is continuous at points (x, y) where $y \neq 0$.
- Check the continuity at points $(a, 0)$ for $a \neq 0$, where the limit presents no indeterminate form.
- Check that the limit of the function as $(x, y) \to (0,0)$ is equal to zero.
- Conclude that the continuity domain is $\mathbb{R}^2$.

b.

- Compute the partial derivatives $f'_x(0,0) = f'_y(0,\ 0) = 0$.
- Compute the remainder $R(x, y)\ =\ f(x, y) - f(0,0) - f'_x(0,0)x - f'_y(0,0)y = f(x, y)$.
- Check that the limit of $\frac{R(x,y)}{||(x,y)||}$ when $(x, y) \to (0,0)$ does not exist by, for instance, computing the directional limits.
- Conclude that the function is not differentiable at the origin.

c.

- Compute the limit of $\frac{f(2+t,-t)-f(2,0)}{t}$ when $t \to 0$, separating the cases $t \to 0^+$ and $t \to 0^-$.
- Conclude that $f'_{(1,-1)}(2,0) = 0$.

d.

- Justify that the function is continuous, and so Weierstrass' theorem is applicable if A is compact.
- Justify that A is compact when $B_1(0,0) \subseteq [1-k, 1+k]^2$.
- Check that this holds precisely when $k \geq 2$.

4.

a.

- Compute $\lim_{x\to 0}\left(\lim_{y\to 0}\frac{e^{x+y^2}-1}{x+y}\right)=\lim_{x\to 0}\frac{e^x-1}{x}=1$ (standard limit)
  - Alternatively: $\lim_{x\to 0}\left(\lim_{y\to 0}\frac{e^{x+y^2}-1}{x+y}\right)=\lim_{x\to 0}\frac{e^x-1}{x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x}{1}=1$ (Cauchy Rule)
- Compute $\lim_{y\to 0}\left(\lim_{x\to 0}\frac{e^{x+y^2}-1}{x+y}\right)=\lim_{y\to 0}\frac{e^{y^2}-1}{y}=\lim_{y\to 0}\frac{e^{y^2}-1}{y^2}\times\lim_{y\to 0}y=\lim_{z\to 0}\frac{e^z-1}{z}\times\lim_{z\to 0}z=1\times 0=0$ (standard limit).
  - Alternatively: $\lim_{y\to 0}\left(\lim_{x\to 0}\frac{e^{x+y^2}-1}{x+y}\right)=\lim_{y\to 0}\frac{e^{y^2}-1}{y}=\left(\frac{0}{0}\right)=\lim_{y\to 0}\frac{2ye^{y^2}}{1}=0\times 1=0$ (Cauchy Rule)

b.

- Compute $\lim_{\substack{(x,y)\to(0,0)\\ y=x}}\frac{e^{x+y^2}-1}{x+y}=\lim_{x\to 0}\frac{e^{x+x^2}-1}{2x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(1+2x)e^{x+x^2}}{2}=\frac{1}{2}$ (Cauchy Rule)

c.

- State that since the iterated limits of f at $(0,0)$ exist and are different, f has no limit at $(0,0)$. Therefore, we cannot define a continuous extension of f at $(0,0)$, and consequently we cannot define a continuous extension of g to $(0,0)$.

d.

- State that since $(0,0)$ is an accumulation point of D_g, the existence of a limit at $(0,0)$ is a necessary condition for the continuity of g at $(0,0)$. If the limit of f at $(0,0)$ does not exist, the limit of g at $(0,0)$ also does not exist, therefore g is not continuous $(0,0)$. Since differentiability implies continuity, if g is not continuous at $(0,0)$, then it is not differentiable at $(0,0)$.

5.

a.

- State that since f is a C^2 function, it is also a C^1 function. And C^1 functions are differentiable.
- State that since f is a C^2 function, f'_x and f'_y are both C^1 functions, therefore both differentiable.

b.

- State that since f is differentiable on $\mathbb{R}^2$, it is also differentiable at $(1,0)$. Given that f is differentiable at (1,0), we get $f'_{(2,3)}(1,0)=\nabla f(1,0)\cdot(2,3)=\left(1,f'_y(1,0)\right)\cdot(2,3)=2+3f'_y(1,0)$
- State that since $f'_{(2,3)}(1,0)=8$, we obtain $2+3f'_y(1,0)=8$, therefore $f'_y(1,0)=2$.
- State that $\nabla f(1,0)=\left(f'_x(1,0),f'_y(1,0)\right)=(1,2)$.

c.

- State that since both f'_x and f'_y are differentiable at $(1,0)$, Schwarz-Young Theorem ensures that $f''_{xy}(1,0) = f''_{yx}(1,0)$, hence $f''_{xy}(1,0) = -1$.

d.

- Analyse the antecedent:
  - if $f'_{(a,b)}(1,0) = -1$, and since f is differentiable at $(1,0)$ we obtain: $f'_{(a,b)}(1,0) = \nabla f(1,0) \cdot (a,b) = af'_x(1,0) + bf'_y(1,0) = a + 2b$, so that $a + 2b = -1$.
  - if $\|(a,b)\| = 1$, then $\sqrt{a^2 + b^2} = 1$, therefore $a^2 + b^2 = 1$.
  - Since we are presented with a conjunction, we must have $\begin{cases} a + 2b = -1 \\ a^2 + b^2 = 1 \end{cases}$. Solving by substitution, we obtain $(a,b) = (-1,0)$ or $(a,b) = \left(\frac{3}{5}, -\frac{4}{5}\right)$.
- State that the antecedent is also true in case that $(a,b) = \left(\frac{3}{5}, -\frac{4}{5}\right)$, in which case the consequent is false. The antecedent may be true and the consequent false, therefore the implication presented is false.