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Midterm

- **Date:** Abril 8, 2022
- Duration: 2 hours
- **Instructions: 1:** The midterm has **four questions. 2:** Write your number and **absolutely nothing else** on this test paper, and **hand it in at the end. 3:** Write your answers on the answer booklet, using the **front and back** of each sheet, **stating** the question you are answering, **never** answering **more than one question on the same sheet**, and **not unstapling** any sheets. **4:** If you want to use any sheet of the answer booklet as space for **drafts**, state it clearly on the **space for the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7:** If, in question **4**, you answer all parts, **only the first five you answer will be graded. 8: No** individual questions about the exam will be answered. **9:** Break a leg (not literally)!

N°: Ignore exercises marked with an × or crossed out.

1. **(4.5 pts)** Consider the function $f: D \subset \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = \frac{x^2y}{\sqrt{x^2+y^2}}$. Consider sets A and B, defined respectively by:

$$A = \{(x,y) \in \mathbb{R}^2 : x = 2 \vee (x,y) \in B_1(1,1)\} \qquad B = A \cup D^c$$

a. **(1 pt)** Represent geometrically set B.

b. **(1 pt)** Define the interior, the boundary, the closure, and the derived set of B.

c. **(1.5 pts)** State, and justify, whether B is open, closed, bounded, compact ~~and connected. State and justify whether A is connected.~~

d. **(1 pt)** Using the definition of limit of a function at a point, show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

2. **(5.5 pts)** Consider the function $f: D_f \subset \mathbb{R}^2 \to \mathbb{R}_0^+ \times (\mathbb{R} \setminus \{0\})$ defined by:

$$f(x,y) = (f_1(x,y), f_2(x,y)) = \left(\sqrt{\frac{x}{x+y}}, x+y\right)$$

a. **(1.25 pts)** Define and geometrically represent D_f.

b. **(1.25 pts)** Define the general level curve of f_1, and geometrically represent, if possible, the one of level 2 and the one which contains $(1,0)$.

c. **(1.5 pts)** Knowing that f is invertible, characterize its inverse function.

d. **(1.5 pts)** Consider the function $g: D_g \subset \mathbb{R}^2 \to \mathbb{R}$, defined by $g(s,t) = \frac{t}{s^2}$. Characterize, if they exist, $(g \circ f)$ and $(f \circ g)$.

3. **(5 pts)** Consider the sequences $(u_n), (v_n), (w_n)$ and (z_n) with general terms:

- $u_n = \frac{1}{kn}, k < 0$
- $v_n = (1 + |u_n|)^n$
- $w_n = (v_n)^{\frac{1}{n^2}}$
- $z_n = (n \cdot u_n)^n$

× **a.** **(0.75 pts)** Study (u_n) regarding monotonicity.

× **b.** **(1 pt)** Find $\lim u_n$, and prove it using the definition of limit of a convergent sequence.

× **c.** **(1.25 pts)** Compute $\lim(v_n + w_n)$.

d. **(1 pt)** Define the set of values of k for which the series $\sum_{n=1}^{+\infty} z_n$ is convergent. For these values, find, as a function of k, the sum of the series.

e. **(1 pt)** Let $A = \left\{\left(\frac{2}{n}, \frac{1}{4n}, \frac{1}{bn}\right) \in \mathbb{R}^3 : n \in \mathbb{N}, b \in \mathbb{R}^+\right\}$. Find the value of b which ensures that the maximum distance between elements of A and the only accumulation point of A is $\sqrt{5}$.

4. **(5 pts)** State and justify the truth value of **five** of the following six propositions:

× **a.** **(1 pt)** Let S be a non-empty set. Then, $\#(P) = 2$ is a necessary condition for P to be a partition of S.

b. **(1 pt)** The set $S = \{x \in \mathbb{R} : x^2 < 4\}$ has an infimum, but not a maximum.

× **c.** **(1 pt)** Let (u_n) and (v_n) be two sequences such that (u_n) is strictly decreasing and that $v_n \leq u_n$ for all $n \in \mathbb{N}$. Then, (v_n) is a decreasing sequence.

d. **(1 pt)** Let $A = \{x \in \mathbb{R} : |2x - 3| \geq 4\}$. Then, B being a subset of A is a sufficient condition for B to be bounded above.

e. **(1 pt)** Consider a sequence (a_n), and the series $\sum_{n=1}^{+\infty} a_n$. If (a_n) is convergent, then the series $\sum_{n=1}^{+\infty} a_n$ converges.

f. **(1 pt)** Let $A = (\,]1,2[\, \cup\,]3,4[) \times]-1,1[$ and $B = [2,3] \times]1,3[$. Then, the points $(2,1)$ and $(3,3)$ are both accumulation points of the set $(A \cup B)$.

Solution Topics

1.

a.

- Find $D = \mathbb{R}^2 \setminus \{(0,0)\}$. Therefore, $D^c = \{(0,0)\}$.
- Correctly represent set B: an open ball centered at $(1,1)$ with radius 1, a vertical line defined by the expression $x = 2$, and an isolated point at $(0,0)$.

b.

- $int(A) = B_1(1,1)$
- $fr(B) = \partial B = \{(x,y) \in \mathbb{R}^2 : \; [(x-1)^2 + (y-1)^2 = 1] \vee (x = 2) \vee (x = \wedge\, y = 0)\}$
- $\bar{B} = \{(x,y) \in \mathbb{R}^2 : \; [(x-1)^2 + (y-1)^2 \leq 1] \vee (x = 2) \vee (x = 0 \; \wedge y = 0)\}$
- $B' = \{(x,y) \in \mathbb{R}^2 : \; [(x-1)^2 + (y-1)^2 \leq 1] \vee (x = 2)\}$

c.

- Show that $B \neq int(B)$, therefore B is not open.
- Show that $B \neq \bar{B}$, therefore B is not closed.
- Show that B is not bounded, since the set $\{(2,y) : y \in \; \mathbb{R}\}$ belongs to B, and there is no ball in $\mathbb{R}^2$ that contains this set.
- Explain that a compact set is a set that is both bounded and closed, and that since B is neither, B is not compact.
- Show that B is not connected, since, for example, if we consider $B = A \cup D^c$, then we have that $\bar{A} \cap D^c = \emptyset$ and that $A \cap \overline{D^c} = \emptyset$. Therefore, B is disconnected.
- Show that A is connected, since it is path-connected: any two points belonging to A may be connected by a curve totally contained in A (that, eventually, goes through $(2,1)$, a point belonging to A).

d.

- Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ using the definition of limit of a function at a point, by finding an example of a distance ε of the inputs to $(0,0)$ (for example, $\varepsilon \leq \sqrt{\delta}$) that ensures the distance of the images to 0 to be smaller than δ, for each δ.

2.

a.

- Show that $D_f = \{(x,y) \in \mathbb{R}^2 : x + y \neq 0 \wedge \frac{x}{(x+y)} \geq 0\} = \{(x,y) \in \mathbb{R}^2 : (x \geq 0 \wedge \; y > -x) \vee (x \leq 0 \; \vee y < -x)\}$.
- Geometrically represent D_f, that is, both the region defined by $x \geq 0$ and $y > -x$, and the region defined by $x \leq 0$ and $y < -x$. Note that neither the line $y = -x$, nor the origin of the referential

belong to D_f.

b.

- Describe the general level set $L_{f_1}^k = \{(x,y) \in D_f : f_1(x,y) = k,\ k \geq 0\} = \left\{(x,y) \in D_f : \sqrt{\frac{x}{x+y}} = k,\ k \geq 0\right\} = \left\{(x,y) \in D_f : \left(y = \frac{1-k^2}{k^2}\,x\,,\ k > 0\right) \vee (x = 0,\ k = 0)\right\}$.
- Represent the level curve with level 2, $L_{f_1}^2 = \left\{(x,y) \in D_f : y = -\frac{3}{4}x\right\}$, respecting D_f $\left((0,0) \notin D_f\right)$.
- Substitute the point $(1,0)$ on the general level curve of f_1 and find $k = 1$, (note that $k = \pm 1 \wedge k \geq 0$). Represent, respecting D_f, $L_{f_1}^1 = \{(x,y) \in D_f : y = 0\}$.

c.

- State that if f is invertible, then $\begin{cases} u = \sqrt{\frac{x}{x+y}} \\ v = x + y \end{cases} \Leftrightarrow \begin{cases} u^2 = \frac{x}{v} \\ y = v - x \end{cases} \Leftrightarrow \begin{cases} x = u^2 v \\ y = v - u^2 v \end{cases}$
- Characterize the inverse function: $f^{-1} : R_f \to D_f\,, (u,v) \longmapsto (u^2 v,\ (1-u^2)v)$.

d.

- State that the dimensions of the codomain of g and those of the domain of f are not compatible, therefore the composition $(f \circ g)$ does not exist.
- State that the dimensions of the codomain of f and those of the domain of g are not compatible, therefore the composition $(g \circ f)$ may exist.
- Describe the domain of $(g \circ f)$: $D_{g \circ f} = \{(x,y) \in \mathbb{R}^2 :\ (x,y) \in D_f \ \wedge \ f(x,y) \in D_g\}$
  - Determine $D_g = \{(s,t) \in \mathbb{R}^2 :\ s^2 \neq 0\} = \mathbb{R} \setminus \{0\} \times \mathbb{R}$,
  - Derive that $f(x,y) \in D_g \Leftrightarrow f_1(x,y) \neq 0 \Leftrightarrow \sqrt{\frac{x}{x+y}} \neq 0 \Leftrightarrow x \neq 0$.
  - Conclude that $D_{g \circ f} = D_f \setminus \{x = 0\}$
- Obtain the analytical expression of the composite function:

$$(g \circ f)(x,y) = g\big(f(x,y)\big) = g\left(\sqrt{\frac{x}{x+y}}\,,\qquad x + y\right) = \frac{(\,x+y)^2}{x}$$

- Characterize the composite function: $(g \circ f) : D_f \setminus \{x = 0\} \to \mathbb{R} \setminus \{0\}\,, (x,y) \longmapsto \frac{(x+y)^2}{x}$

3.

a.

- Show that $u_{n+1} - u_n = -\frac{1}{kn(n+1)}$.
- Justify that $-\frac{1}{kn(n+1)} > 0, \forall n \in \mathbb{N}$, since $k < 0$. Therefore, (u_n) is a strictly increasing sequence.

b.

- Show that $\lim u_n = 0$.
- Prove, using the limit definition, that $u_n \to 0$.

c.

- State that $\lim v_n = \lim\left(1+\frac{1}{|kn|}\right)^n = \lim\left(1+\frac{\frac{1}{|k|}}{n}\right)^n = e^{\frac{1}{|k|}}$.
- State that $\lim w_n = \lim\left[\left(1+\frac{1}{|kn|}\right)^n\right]^{\frac{1}{n^2}} = \lim\left(1+\frac{1}{|kn|}\right)^{\frac{1}{n}} = 1^0 = 1$.
- Conclude that $\lim(v_n + w_n) = e^{\frac{1}{|k|}} + 1$.

d.

- State that $\sum_{n=1}^{+\infty} z_n = \sum_{n=1}^{+\infty}(n \cdot u_n)^n = \sum_{n=1}^{+\infty}\left(\frac{1}{k}\right)^n$.
- State that $\sum_{n=1}^{+\infty}\left(\frac{1}{k}\right)^n$ is a geometric series and that $-1 < \frac{1}{k} < 0$ forces the series to be convergent (note that $k < 0$, therefore $\frac{1}{k} < 0$).
- Conclude that $k < -1$.

e.

- Show that $(0,0,0)$ is the only accumulation point of set A.
- Write the condition $\sqrt{2^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{b}\right)^2} = \sqrt{5}$.
- Conclude that $b = \frac{4}{\sqrt{5}}$.

4.

a. State that the proposition is false, since, for example, if $S = \{1,2,3\}$ and $P = \{\{1\},\{2\},\{3\}\}$, then P is a partition of S (since it is formed by disjoint sets of elements of S whose union is S, but $\#(P) = 3$.

b. State that the proposition is true, since if we consider a set $S =]-2,2[$, the *infimum* exists and is (-2), but the *supremum*, which is 2, does not belong to the set. Therefore, S does not have a maximum.

c. State that the proposition is false since, for example, if $u_n = \frac{1}{n}$ (which is strictly decreasing), and $v_n = -\frac{1}{n}$, then $v_n < 0 < u_n\ \forall n \in \mathbb{N}$, but v_n is a strictly increasing sequence.

d. State that the proposition is false since $A = \left]-\infty, -\frac{1}{2}\right] \cup \left[\frac{7}{2}, +\infty\right[$, and for example the set $B = [4, +\infty[$ is a subset of A, but is not bounded above.

e. State that the proposition is false since, for example, $a_n = 2 + \frac{1}{n}$ is a convergent sequence

$(\lim a_n = 2)$, but is not a null sequence. Then, following the General Convergence Theorem for series, the series $\sum_{n\geq 1} a_n$ cannot be convergent.

f. State that the proposition is true since given that the derived set of $A \cup B$ is $(A \cup B)' = \big(([1,2] \cup [3,4]) \times [-1,1]\big) \cup ([2,3] \times [1,3])$, then both points $(2,1) \in (A \cup B)'$ and $(3,3) \in (A \cup B)'$ are accumulation points of $A \cup B$.