Nova School of Business and Economics
Calculus I 2021 – 2022 S2
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Resit Exam

- **Date:** June 20th, 2022
- **Duration:** 2 hours and 30 minutes
- **Instructions: 1:** The exam has **four questions. 2:** Write your number and **absolutely nothing else** on this exam paper, and **hand it in at the end. 3:** Write your answers on the answer booklet, using the **front and back** of each sheet, **stating** the question you are answering, **never** answering **more than one question on the same sheet**, and **not unstapling** any sheets. **4:** If you want to use any sheet of the answer booklet as space for **drafts**, state it clearly on the **space for the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7: No** individual questions about the exam will be answered.

Nº: Ignore exercises marked with an × or crossed out.

1. (7 pts) Let $f: D \subset \mathbb{R}^2 \to R_f \subset \mathbb{R}^2$ and $g: D_g \subset \mathbb{R}^2 \to \mathbb{R}$ be the functions defined by:

$$f(x,y) = (\, f_1(x,y), f_2(x,y)\,) = \left(\frac{1}{\sqrt{y - x^2 + 1}}, \frac{1}{x}\right) \qquad g(x,y) = \frac{2x^3 - yx^2}{\sqrt{x^2 + y^2} + \ln(1 + |xy|)}$$

a. (1 pt) Define and geometrically represent D.

b. (1.25 pts) Define the boundary and the derived set of D. Is D open? ~~Is D connected~~? Justify.

c. (1.25 pts) Define the range of f, R_f. Define the general level curve of f_1, and geometrically represent, if possible, the level curves for the levels $-1, \frac{1}{2}$ and 1.

d. (2 pts) State, and justify, whether f is invertible. If so, characterize f^{-1}.

e. (1.5 pts) Let $h: D_h \subset \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $h(x,y) = g(x,y) \times f_2(x,y)$. State, and justify, whether there exists a continuous extension of h to $(0,0)$.

2. (6 pts) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the function defined by:

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{\sqrt{x^2 + y^2}} & , \quad x > 0 \\ \ln(1 + x^2 + y^2) & , \quad x \leq 0 \end{cases}$$

a. (2 pts) Define the continuity domain of f.

b. (1 pt) Show that $\nabla f(0,0) = (0,0)$.

c. (2 pts) State and justify whether f is differentiable at $(0,0)$.

d. (1 pt) Compute, if possible, $f'_{(10,5)}(0,0)$.

3. **(5 pts)** Consider the sequences (p_n), (v_n), (w_n) and the sets A, P and C, defined by:

$$p_n = \frac{(1-x)^n}{3^{n+1}} \qquad A = \left\{ x \in \mathbb{R} : x - 2 > \frac{x^2 - 8}{x} \right\}$$

$$v_n = 1 + (n!)^{-n} \qquad P = \left\{ x \in \mathbb{R} : \sum_{n=1}^{+\infty} p_n \text{ is convergent} \right\}$$

$$w_n = \left(1 + \frac{3}{n^2+1}\right)^{n^2-1} \qquad C = \left\{ x \in \mathbb{R} : \left(x = \lim v_n\right) \vee \left(x = \lim w_n\right) \right\}$$

a. **(1 pt)** Show that $A = B_2(2)$.

b. **(1 pt)** Comment on the truth value of the following proposition:

"$x \notin P$ is a necessary condition for $x \in A^c$".

c. **(1.5 pts)** Find the value of x such that $\sum_{n=1}^{+\infty} p_n = \frac{1}{6}$.

d. **(1.5 pts)** Show that the cardinal number of the set $(A \cap C)$ is 1.

4. **(2 pts)** State and justify the truth value of the following propositions:

a. **(1 pt)** Let $f: [-2,2]^2 \to \mathbb{R}$ be a C^1 function such that $\nabla f(1,1) = (0,-1)$. Let $g: [-2,2]^2 \to \mathbb{R}$ be defined by $g(x,y) = 1 - 2f(x,y)$, and $w: \{(u,v) \in \mathbb{R}^2 : \|(u,v)\| = 1\} \to \mathbb{R}$ be defined by $w(u,v) = g'_{(u,v)}(1,1)$. Then, the global maximum of w occurs at $(u,v) = (0,-1)$.

b. **(1 pt)** Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function. Let $g: \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ be defined by the expression $g(u,v) = f'_{(u,v)}(a,b)$, where $(a,b) \in \mathbb{R}^2$ and $\|(a,b)\| \neq 0$. Then, $\frac{g(1,2)}{g(3,2)} < 0 \Rightarrow \exists (u,v) \in D_g : g(u,v) = 0$.

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Solution Topics

1.

a.

- Indicate that $D = D_f = \{(x, y) \in \mathbb{R}^2 : y > x^2 - 1 \ \wedge x \neq 0\}$.
- Represent D_f, which corresponds to the region inside the parabola of vertex $(0, -1)$, excluding the axis Oy.

b.

- Indicate that $fr(D) = \{(x, y) \in \mathbb{R}^2 : y = x^2 - 1 \ \vee (y > x^2 - 1 \ \wedge x = 0)\}$.
- Indicate that $D' = \{(x, y) \in \mathbb{R}^2 : y \geq x^2 - 1\}$.
- Indicate that $int(D) = \{(x, y) \in \mathbb{R}^2 : y > x^2 - 1 \ \wedge x \neq 0\} = D$, thus D is open.
- Justify that D is not connected because it is possible to find two subsets, $D_1 = \{(x, y) \in \mathbb{R}^2 : y > x^2 - 1 \ \wedge x > 0\}$ and $D_2 = \{(x, y) \in \mathbb{R}^2 : y > x^2 - 1 \ \wedge x < 0\}$, such that $D = D_1 \cup D_2$ but $\overline{D_1} \cap D_2 = \emptyset$ and $\overline{D_2} \cap D_1 = \emptyset$.

c.

- Indicate that $R_f = \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\})$ since the value that f_2 takes at a specific point $(x, y) \in D$ does not restrict the value that f_1 may take at that point.
- Indicate that $L_{f_1}^k = \left\{(x, y) \in D : \frac{1}{\sqrt{y - x^2 + 1}} = k\ ,\ k > 0\right\}$, that is, $L_{f_1}^k = \left\{(x, y) \in D_f : \mathrm{y} = \mathrm{x}^2 + \frac{1}{k^2} - 1\ , k > 0\right\}$.
- Indicate that $\nexists L_{f_1}^{-1}$ because $-1 \notin R_{f_1}$.
- Indicate that $L_{f_2}^{1/2} = \left\{(x, y) \in D_f : y = x^2 + 3\right\}$, and represent the level curve, corresponding the parabola with the exclusion of point $(0,3)$.
- Indicate that $L_{f_2}^{1} = \left\{(x, y) \in D_f : y = x^2\right\}$, and represent the level curve, corresponding to the parabola except for the point $(0,0)$.

d.

- Justify that f is surjective since $CD_f = R_f$.
- Show that $f(x_1, y_1) = f(x_2, y_2) \Rightarrow (x_1, y_1) = (x_2, y_2)$, thus f is injective.
- Justify that f is invertible since it is bijective.
- Define function $f^{-1} : R_f \to D\ , f^{-1}(u, v) = \left(\frac{1}{v}, \frac{1}{v^2} + \frac{1}{u^2} - 1\right)$.

e.

- Define function $h(x) = \frac{2x^2 - yx}{\sqrt{x^2 + y^2} + \ln(1 + |xy|)}$.
- Recognize that $D_h = D_f \setminus \{(0,0)\}$ thus $(0,0) \in D_h{}'$.
- Prove, using the definition, that $\lim\limits_{(x,y) \to (0,0)} h(x, y) = 0$.
- Conclude that since the limit of the function at $(0,0)$ is finite, we can define a continuous extension of h at that point.

2.

a.

- Justify that f is continuous for $x > 0$ and for $x < 0$.
- Study the continuity of f for points $(x, y) = (0, b), b \in \mathbb{R}$.
- If $b \neq 0$:
    - $f(0, b) = \lim\limits_{\substack{(x,y)\to(0,b)\\ x<0}} f(x, y) = \lim\limits_{\substack{(x,y)\to(0,b)\\ x<0}} \ln(1 + x^2 + y^2) = \ln(1 + b^2)$
    - $\lim\limits_{\substack{(x,y)\to(0,b)\\ x>0}} f(x, y) = \lim\limits_{\substack{(x,y)\to(0,b)\\ x>0}} \frac{x^2 y}{\sqrt{x^2+y^2}} = \frac{0}{b} = 0$
    - Conclude that f is not continuous at points $(0, b)$, $b \neq 0$.
- If $b = 0$:
    - $f(0,0) = \lim\limits_{\substack{(x,y)\to(0,0)\\ x<0}} f(x, y) = \lim\limits_{\substack{(x,y)\to(0,0)\\ x<0}} \ln(1 + x^2 + y^2) = \ln(1) = 0$
    - $\lim\limits_{\substack{(x,y)\to(0,0)\\ x>0}} f(x, y) = \lim\limits_{\substack{(x,y)\to(0,0)\\ x>0}} \frac{x^2 y}{\sqrt{x^2+y^2}} = 0$, using the definition of limit of a function, finding a distance on the inputs space for each given distance δ in the outputs space, for example, $\sqrt{\delta}$.
    - Conclude that f is continuous at $(0, 0)$
- State that the continuity domain of f is $\mathbb{R}^2 \setminus \{(0, b) \in \mathbb{R}^2 : b \neq 0\}$.

b.

- State that $\nabla_f(0,0) = \left(f_x'(0,0), f_y'(0,0)\right)$.
- Show that $f_x'(0,0) = 0$ using the definition of partial derivative.
- Show that $f_y'(0,0) = 0$ using derivation rules, or the definition of partial derivative.
- Conclude that $\nabla_f(0,0) = (0,0)$.

c.

- State that f is differentiable at $(0,0)$ if, in a neighbourhood of $(0,0)$, it can be written as:
  $f(x, y) = f(0,0) + f_x'(0,0)x + f_y'(0,0)y + R(x, y) = 0 + 0 + 0 + R(x, y)$, such that
$$\lim_{(x,y)\to(0,0)} \frac{R(x, y)}{\|(x, y)\|} = \lim_{(x,y)\to(0,0)} \frac{f(x, y)}{\|(x, y)\|} = 0$$
- Show, using the definition, or a change of variable $z = x^2 + y^2$ and then using the Cauchy rule or the resulting standard limit, that:
$$\lim_{\substack{(x,y)\to(0,0)\\ x<0}} \frac{f(x, y)}{\|(x, y)\|} = \lim_{(x,y)\to(0,0)} \frac{\ln(1 + x^2 + y^2)}{\sqrt{x^2 + y^2}} = 0$$
- Show, using the definition, that:
$$\lim_{\substack{(x,y)\to(0,0)\\ x>0}} \frac{f(x, y)}{\|(x, y)\|} = \lim_{\substack{(x,y)\to(0,0)\\ x>0}} \frac{\frac{x^2 y}{\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}} = \lim_{\substack{(x,y)\to(0,0)\\ x>0}} \frac{x^2 y}{x^2 + y^2} = 0$$

d.

State that since f is differentiable at $(0,0)$, the directional derivative can be computed as

$$f'_{(10,5)}(0,0) = \nabla f(0,0) \cdot (10,5) = (0,0) \cdot (10,5) = 0 + 0 = 0$$

3.

a.

- Solve the inequality given, by solving $x - 2 - \frac{x^2-8}{x} > 0$, and NOT $x^2 - 2x > x^2 - 8$.
    - Solve $\frac{8-2x}{x} > 0$, by solving $(8 - 2x > 0 \wedge x > 0) \vee (8 - 2x < 0 \wedge x < 0)$, or using a sign diagram.
    - Conclude that $A =]0,4[= B_2(2)$.

b.

- State that if "$x \notin P$ is a necessary condition for $x \in A^c$", then $(x \in A^c) \Rightarrow (x \notin P)$.
- State that the contrapositive of the statement above is $(x \in P) \Rightarrow (x \in A)$.
- State that $\sum_{n=1}^{+\infty} p_n$ is a geometric series with common ratio $r = \left(\frac{1-x}{3}\right)$, convergent if $\left|\frac{1-x}{3}\right| < 1$, that is, if $x \in]-2,4[$.
- State that the implication $(x \in]-2,4[) \Rightarrow (x \in]0,4[\,)$ is false. For example, if $x = -1$, the antecedent is true, but the consequent is false.

c.

- State that if the sum of the geometric series is $\frac{1}{6}$, then the series is convergent, and its sum can be computed by $S = p_1 \times \frac{1}{1-r} = \frac{1-x}{9} \times \frac{1}{1-\frac{1-x}{3}} = \frac{1-x}{6-3x}$.
- Solve $\frac{1-x}{6-3x} = \frac{1}{6} \Leftrightarrow x = 0$.
- Conclude that $x = 0$ is the value of x for which $\sum_{n=1}^{+\infty} p_n = \frac{1}{6}$.

d.

- Compute $\lim v_n = \lim 1 + (n!)^{-n} = \lim 1 + \frac{1}{(n!)^n} = 1$
- Compute $\lim w_n = \left(1 + \frac{3}{n^2+1}\right)^{n^2-1} = \lim \left(1 + \frac{3}{n^2+1}\right)^{n^2+1} \times \lim \left(1 + \frac{3}{n^2+1}\right)^{-2} = e^3$.
- State that $C = \{1, e^3\}$, therefore $(A \cap C) =]0,4[\cap \{1, e^3\} = \{1\}$, given that $e^3 > 4$.
- State that therefore $\#(A \cap C) = 1$.

4.

a.

- State that if f is C^1, then f is differentiable in D_f.
- State that g is differentiable in D_f since it results from elementary operations applied to f.
- Using one of the properties of differentiable functions, conclude that, since g is differentiable at $(1,1) \in [-2,2]^2$, the maximum of w occurs when $(u,v) = \frac{\nabla g(1,1)}{||\nabla g(1,1)||} = \frac{-2\nabla f(1,1)}{||-2\nabla f(1,1)||} = \frac{-2(0,-1)}{||-2(0,-1)||} = \frac{(0,2)}{||(0,2)||} = (0,1)$
- Conclude that $(u,v) = (0,1) \neq (0,-1)$, therefore the proposition is false.

b.

- State that if f is C^1, then f is differentiable in $D_f = \mathbb{R}^2$.
- State that if f is differentiable at $(a,b) \in \mathbb{R}^2$, then $g(u,v) = f'_{(u,v)}(a,b) = f'_x(a,b).u + f'_y(a,b).v$.
- State that the function defined by $g(u,v) = f'_x(a,b).u + f'_y(a,b).v$ is continuous at $(u,v) \in \mathbb{R}^2 \backslash \{(0,0)\}$ since it is a polynomial function of degree 1.
- Use the Intermediate Value Theorem (Bolzano´s Theorem) to conclude that there is a vector $(u,v) \in D_g$ such that $g(u,v) = 0$, since g is continuous and $g(1,2) \times g(3,2) < 0$. The proposition is true.

Note: it was also possible to create a new function, $h: \mathbb{R} \to \mathbb{R}$, defined by $h(u) = g(u,2)$, and apply the Intermediate Value Theorem to h in the interval $[1,3]$. Since h is continuous in $[1,3]$ and $h(1) \times h(3) < 0$, then $\exists c \in]1,3[: h(c) = g(c,2) = 0$.