Nova School of Business and Economics
Cálculo I 2021 – 2022 S2
Helena Almeida **João Farinha**
Jessica Lomba **Pamela Pacciani**

Regular Exam

- **Date:** May 20, 2022
- **Duration:** 2 hours and 30 minutes
- **Instructions: 1:** The exam has **four questions. 2:** Write your number and **absolutely nothing else** on this exam paper, and **hand it in at the end. 3:** Write your answers on the answer booklet, using the **front and back** of each sheet, **stating** the question you are answering, **never** answering **more than one question on the same sheet,** and **not unstapling** any sheets. **4:** If you want to use any sheet of the answer booklet as space for **drafts,** state it clearly on the **space for the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7:** If, in question **4**, you answer all parts, **only the first four you answer will be graded. 8: No** individual questions about the exam will be answered. **9:** Break a leg (not literally)!

Nº: Ignore exercises marked with an × or crossed out.

1. **(6 pts)** Let $f\colon D_f \subset \mathbb{R}^2 \to \mathbb{R}^2$ and $g\colon D_g \subset \mathbb{R}^2 \to \mathbb{R}$ be functions defined by:

$$f(x,y) = \left(\sqrt{xy}\ ,\ e^{xy}\right) \qquad\qquad g(s,t) = (ln\,t)(ln\,s)$$

a. **(0,5 pts)** Define and geometrically represent the domain of f.

b. **(1,5 pts)** Define the general level curve of f_2, and geometrically represent, if possible, the one of level e^2 and the one which contains $(5{,}0)$.

c. **(1,5 pts)** Show that the function $(g \circ f)$ is defined by $(g \circ f)(x,y) = \frac{1}{2}(xy)\ln(xy)$. Compute $\nabla_{g\circ f}(1{,}1)$ and, based on this result, compute $(g \circ f)'_{(2,2)}(1{,}1)$.

d. **(1,5 pts)** Consider the function defined by $h(x) = (g \circ f)(x,x)$. Show that we can define a function $\tilde{h}$ that is the result of a continuous extension of h to $x = 0$.

e. **(1 pt)** Show that the equation $\tilde{h}(x) = e^x - 2$ has at least one solution on the interval $]0{,}1[$.

2. **(5 pts)** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function defined by:

$$f(x,y) = \begin{cases} y\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if} \quad y \neq x \\ 0 & \text{if} \quad y = x \end{cases}$$

a. **(2 pts)** Define the continuity domain of f.

b. **(1 pt)** Show that $\nabla_f(0{,}0) = (0, -1)$.

c. **(2 pts)** Study f regarding differentiability at $(0{,}0)$.

3. **(5 pts)** Consider the sequence (u_n) with general term $u_n = \frac{n+1}{n}$, and the following sets:

$$A = \{(u_n, u_n) \in \mathbb{R}^2 : n \in \mathbb{N}\} \qquad B = \{(-u_n, -u_n) \in \mathbb{R}^2 : n \in \mathbb{N}\}$$
$$C =]-1,1]^2 \qquad D = A \cup B \cup C$$

a. **(1,5 pts)** Find the interior, the boundary, and the derived set of D. Is D compact? Is ~~D connected~~? Justify.

b. **(2,5 pts)** Let (v_n) and (w_n) be the sequences defined by the general terms $v_n = (u_n)^{-n}$ and $w_n = \sqrt[n]{(n+1)! - n!}$.

(i) **(1,5 pts)** Compute, if it exists, $\lim (v_n + w_n)$.

(ii) **(1 pt)** Let (b_n) be a sequence defined by $b_n = \begin{cases} v_n & , n \text{ odd} \\ w_n & , n \text{ even} \end{cases}$. Comment on the truth value of the following proposition: "$\forall \delta > 0, \exists m \in \mathbb{N}: \left(\exists n \geq m : \left|b_n - \frac{1}{e}\right| < \delta\right)$".

c. **(1 pt)** Find the second term of the sequence of the partial sums of b_n. Does the series $\sum_{n\geq 1} b_n$ converge? Justify.

4. **(4 pts)** State and justify the truth value of **four** of the following five propositions:

a. **(1 pt)** Consider two non-empty sets A and B. Then, A being path-connected is a necessary condition for $(A \cap B)$ being convex.

b. **(1 pt)** If $(u_n)_{n\in\mathbb{N}}$ is a sequence with general term $u_n = k(-1)^n$, where $k \in \mathbb{R}^+$, then the cardinal number of the set of all sequences $(v_n)_{n\in\mathbb{N}}$ such that the product $(u_n \cdot v_n)_{n\in\mathbb{N}}$ defines a null sequence, is infinite.

c. **(1 pt)** Let $\sum_{n\geq 1} a_n$ be a divergent series, and $\sum_{n\geq 1} b_n$ be a geometric series. Then, $\sum_{n\geq 1}(a_n + b_n)$ is a divergent series.

d. **(1 pt)** Let $f: \mathbb{R} \to \mathbb{R}$ be a class C^2 function, such that f' is an even function, $f'(3) = 0$, and $f'(x) \neq 0$ if $x \in [0,3[$. Then, f'' has at most one zero on $]-3,3[$.

e. **(1 pt)** Let $f: D \subset \mathbb{R}^3 \to \mathbb{R}$ be a function such that $f(0,0,0) = 0$, $f'_x(x,y,z) = y(2x+y)$ and $f'_y(x,y,z) = x(x+2y)$. Then, f is continuous at $(0,0,0)$ if $\lim_{h\to 0} \frac{h}{f(0,0,h)} = 3$.

Solution Topics

1.

a.

- State that $D_f = \{(x,y) \in \mathbb{R}^2 : xy \geq 0\} = \{(x,y) \in \mathbb{R}^2 : (x \geq 0 \wedge y \geq 0) \vee (x \leq 0 \wedge y \leq 0)\}$.
- Represent D_f, corresponding to the first and third quadrants (including the coordinate axis).

b.

- State that $L_{f_2}^k = \{(x,y) \in \mathbb{R}^2 : e^{xy} = k, k > 0\}$, that is, $L_{f_2}^k = \left\{(x,y) \in D_f : xy = \ln k\,, k > 0\right\}$
- State that $L_{f_2}^{e^2} = \left\{(x,y) \in D_f : y = \frac{2}{x}, x \neq 0\right\}$ and represent the level curve.
- Show that the level curve that contains the points $(5,0)$ corresponds to the case when $k = 1$.
- State that $L_{f_2}^1 = \left\{(x,y) \in D_f : xy = 0\right\}$, that is, $L_{f_2}^1 = \{(x,y) \in D_f : (x = 0) \vee (y = 0)\}$, and represent the level curve.

c.

- Show that $(g \circ f)(x,y) = g\big(f(x,y)\big) = g\big(\sqrt{xy}, e^{xy}\big) = \big(\ln\sqrt{xy}\big)(\ln e^{xy}) = \left(\frac{1}{2}\ln(xy)\right)(xy) = \frac{1}{2}(xy)\ln(xy)$.
- State that $(g \circ f)$ is differentiable in its domain, $D_{g\circ f} = \{(x,y) \in \mathbb{R}^2 : xy > 0\}$.
- We can then use the rules of differentiation to compute:
  - $(g \circ f)'_x(x,y) = \frac{1}{2}y\ln(xy) + \frac{1}{2}(xy)\frac{1}{xy}y = \frac{y}{2}(1 + \ln(xy))$, and therefore $(g \circ f)'_x(1,1) = \frac{1}{2}$
  - $(g \circ f)'_y(1,1) = \frac{1}{2}$, using the same procedure.
- State that given that $g \circ f$ is differentiable at $(1,1)$, then $(g \circ f)'_{(2,2)}(1,1)$ can be computed (which is what is requested) as $(g \circ f)'_{(2,2)}(1,1) = \nabla_{g\circ f}(1,1) \cdot (2,2) = \left(\frac{1}{2}, \frac{1}{2}\right) \cdot (2,2) = 2$.

d.

- Find the expression of h, $h(x) = (g \circ f)(x,x) = \frac{1}{2}x^2\ln x^2$.
- Compute $\lim\limits_{x\to 0} h(x) = \lim\limits_{x\to 0}\frac{1}{2}x^2\ln x^2 = (0 \times \infty) = \frac{1}{2}\lim\limits_{x\to 0}\frac{\ln x^2}{\frac{1}{x^2}} = \left(\frac{\infty}{\infty}\right) = \frac{1}{2}\lim\limits_{x\to 0}\frac{\frac{2x}{x^2}}{\frac{-2x}{x^4}} = -\frac{1}{2}\lim\limits_{x\to 0} x^2 = 0$, where the Cauchy rule was used to solve the indeterminate form $\left(\frac{\infty}{\infty}\right)$.
- Describe the function $\tilde{h} : \mathbb{R} \to \mathbb{R}, \tilde{h}(x) = \begin{cases} h(x) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

e.

- Define the function $p(x) = \begin{cases} \frac{1}{2}x^2\ln x^2 - (e^x - 2) & , x \neq 0 \\ 0 - (e^x - 2) & , x = 0 \end{cases}$
- Justify that p is a continuous function on its domain, hence continuous on $[0,1]$.
- Calculate $p(0) = 1 > 0$ and $p(1) = 2 - e < 0$, hence it verifies $p(0) \times p(1) < 0$.
- Justify that since the conditions of the Corollary to the Bolzano Theorem are met, one can conclude that $\exists c \in]0,1[: p(0) = 0$, which implies that $\exists c \in]0,1[: \tilde{h}(c) = e^c - 2$.

2.

a.

- Justify the continuity of f for points such that $y \neq x$.
- State that for points where $y = x$, f is continuous if $\lim_{(x,y)\to(a,a)} f(x,y) = f(a,a) = 0$.
- Show that $a \neq 0$, $\lim_{\substack{(x,y)\to(a,a)\\ y\neq x}} f(x,y) = \lim_{\substack{(x,y)\to(a,a)\\ y=x}} f(x,y) = 0$, hence f is continuous at (a,a).
- Show, using the definition of limit of a function, that $\lim_{\substack{(x,y)\to(0,0)\\ y\neq x}} f(x,y) = 0$ (for example, by obtaining the relationship $\varepsilon \leq \frac{\delta}{2}$).
- Conclude that $\lim_{\substack{(x,y)\to(0,0)\\ y\neq x}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = f(0,0) = 0$.
- Conclude that the continuity domain of f is $\mathbb{R}^2$.

b.

- State that $\nabla_f(0,0) = \left(f'_x(0,0), f'_y(0,0)\right)$.
- Show, using the definition of partial derivative, that $f'_x(0,0) = 0$
- Show, using the definition of partial derivative, that $f'_y(0,0) = -1$
- Conclude that $\nabla_f(0,0) = (0,-1)$

c.

- State that f is differentiable in a neighbourhood of $(0,0)$ if we can write it as:

$f(x,y) = f(0,0) + f'_x(0,0)x + f'_y(0,0)y + R(x,y) = 0 + 0 - y + R(x,y)$, where

$$\lim_{(x,y)\to(0,0)} \frac{R(x,y)}{\|(x,y)\|} = 0$$

- Obtain

$$\lim_{(x,y)\to(0,0)} \frac{R(x,y)}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{f(x,y)+y}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{y\frac{x^2-y^2}{x^2+y^2}+y}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{2x^2y}{\|(x,y)\|^3}$$

- For example, verify that $\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{2x^2y}{\|(x,y)\|^3} = \lim_{x\to0} \frac{2x^3}{\left(\sqrt{2x^2}\right)^3} = \lim_{x\to0} \frac{2x^3}{2\sqrt{2}\,x^2|x|} = \lim_{x\to0} \frac{2x}{2\sqrt{2}|x|} = \pm\frac{1}{\sqrt{2}} \neq 0$.
- Conclude that f is not differentiable at $(0,0)$.

3.

a.

- Represent D or explain how it is composed.
- Show that $int(D) =]-1,1[^2$, $front(D) = A \cup B \cup ([-1,1] \backslash]-1,1[)$ and that $D' = [-1,1]^2$.
- Explain that D is not a closed set $(int(D) \cup front(D) = \overline{D} \neq D)$ therefore it is not a compact set (closed and bounded).
- Explain that D is not a connected set and illustrate with a partition of D, for example $D_1 = \{(-2,2)\}$ and $D_2 = D \backslash D_1$.

b.

(i)

- Compute $\lim\ v_n$ using the Neper´s limit.
- Compute $\lim\ w_n$ using the result about the root of index n of a sequence with nonnegative terms.
- Compute $\lim\ (v_n + w_n)$.

(ii)

- Conclude that the proposition is true. The sequence (b_n) does not have a limit (we can easily find two subsequences of (b_n) with a different limit), but we can find a natural index m such that, for indexes $n \geq m$, there are terms of b_n that are closer and closer to $\frac{1}{e}$. This can happen when n is odd. The proposition is not stating that the limit of (b_n) is $\frac{1}{e}$.

c.

- Show that $S_2 = b_1 + b_2 = \frac{1}{2} + 2 = \frac{5}{2}$.
- Using the necessary condition for convergence, conclude that $\sum_{n\geq 1} b_n$ is a divergent series since $\lim b_n \neq 0$.

4.

a. The statement is false since, for example, if $A = [1,2] \cup [3,4]$ and B= $[0,2]$, then A is not path connected (it is not possible to connect points $2 \in A$ and $3 \in A$ by a curve completely contained in A) but $A \cap B = [1,2]$ is a convex set, since the line segment that connects any two points of $A \cap B$ is completely contained in that set.

b. The statement is true, since as $|u_n| \leq k$, with fixed $k \in \mathbb{R}^+$, then $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence and, therefore, the product sequence $(u_n \cdot v_n)_{n\in\mathbb{N}}$ converges to 0 whenever $(v_n)_{n\in\mathbb{N}}$ converges to 0. As there exist infinitely many sequences $(v_n)_{n\in\mathbb{N}}$ converging to 0 (for example, $v_{n,t} = \frac{t}{n}$, for any $t \in \mathbb{R}$) then $\#\left\{(v_n)_{n\in\mathbb{N}} : \lim v_n = 0\right\} = \infty$.

c. The statement is false since, for example, if $a_n = 1^n = -b_n = -1^n$, then $\lim a_n \neq 0$, by the General Convergence Criterion, the series $\sum_{n\geq1} a_n$ diverges, and as $\frac{b_{n+1}}{b_n} = 1$ (constant ratio), the series $\sum_{n\geq1} b_n$ is a geometric series (also divergent). But $\sum_{n\geq1}(a_n + b_n) = \sum_{n\geq1}(1-1) = \sum_{n\geq1} 0 = 0 \in \mathbb{R}$, thus convergent.

d. The statement is false since, for example, if $f'(x) = x^4 - x^2 - 72$ (polynomial function, thus of class C^1, differentiable and with continuous derivative), then $f'(x)$ is an even function with $f'(3) = f'(-3) = 0$ and $f'(x) \neq 0 \; \forall x \in]-3,3[$. Its derivative is $f''(x) = 4x^3 - 2x$, and we have $f''(x) = 0 \Leftrightarrow x = -\frac{1}{\sqrt{2}} \vee x = 0 \vee x = \frac{1}{\sqrt{2}}$, that is, f'' has more than one zero in the interval $]-3,3[$.

e. The statement is true, since as both derivatives of f given, f'_x e f'_y , are polynomial functions, hence continuous at $(0,0,0)$, and the limit presented corresponds to $3 = \lim_{h\to0} \frac{h}{f(0,0,h)-f(0,0,0)} = \left(\lim_{h\to0} \frac{f(0,0,h)-f(0,0,0)}{h}\right)^{-1} = \frac{1}{f'_z(0,0,0)}$ therefore making $f'_z(0,0,0) = \frac{1}{3} \in \mathbb{R}$. As $D \subset \mathbb{R}^3$, all partial derivatives of f at $(0,0,0)$ exist and in particular $n - 1 = 2$ of them are continuous at that point, thus making f differentiable at $(0,0,0)$ and consequently continuous at that point.