Nova School of Business and Economics
2021 – 2022 S1
Calculus I

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Midterm

- **Date:** November 13, 2021
- Duration: 2 hours
- **Instructions: 1:** The test has **four questions. 2:** Write your number and **absolutely nothing else** in this test paper, and **hand it in at the end. 3:** Answer the test on the answer booklet, using the **front and back** of each sheet, **stating** the question you are answering, **never** answering **more than one question on the same sheet**, and **not unstapling** any sheet. **4:** If you want to use any sheet of the answer booklet as space for **drafts,** state it on the **space for the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7:** If, in question **4**, you answer all parts, **only the first five will be graded. 8: No** individual questions about the test will be answered. **9:** Break a leg (not literally)!

Nº:

1. **(5 pts)** Consider three sets in $\mathbb{R}$, A, B and C, such that:

- $A = \left\{x \in \mathbb{R}: \frac{|2x+1|-1}{x^2+|x|+1} \leq 0\right\}$
- $C = A^c \cup \mathbb{Q}^c$
- $B = A^c \cap \mathbb{R}^-$

a. **(1 pt)** Show that $A = [-1,0]$.

b. **(0.75 pts)** State and justify whether B has a maximum and, if so, identify it.

c. **(1 pt)** Define the interior, the boundary and the derived set of C.

d. **(0.75 pts)** State and justify whether C^c is bounded.

e. **(1.5 pts)** State and justify the truth value of the following proposition: "C is convex ~~or connected.~~".

2. **(6 pts)** Consider the function $f: D_f \subset \mathbb{R}^2 \to R_f \subset \mathbb{R}^2$, defined by:

$$f(x,y) = (f_1(x,y), f_2(x,y)) = \left(\log_2(2y - x), \sqrt{x-y}\right)$$

a. **(1 pt)** Define and geometrically represent D_f.

b. **(1.5 pts)** Knowing that the range of f_1 is $\mathbb{R}$, define the general level curve of f_1, and geometrically represent the one of level 1 and the one which includes $(3,2)$.

c. **(1.5 pts)** Knowing that f is invertible, characterize its inverse.

d. **(1 pt)** Consider the function $g: D_g \subset \mathbb{R}^2 \to \mathbb{R}$, defined by $g(s,t) = \frac{t^4+1}{2^s}$. Characterize, if they exist, $g \circ f$ and $f \circ g$.

e. **(1 pt)** Consider two functions, $h: \mathbb{R} \to \mathbb{R}$ and $i: D_i \subset \mathbb{R}^3 \to R_i \subset \mathbb{R}^3$, such that:

- h is odd
- i is defined by $i(x,y,z) = \left(f_1(x,y), f_2(x,y), \frac{1}{1+h^2(z)}\right)$

Show that i is not invertible.

3. **(4 pts)** Consider $a \in \mathbb{R}$, $b \in \mathbb{R}^2$, three sets in $\mathbb{R}^2$, A, B and C, such that:

- $A = [0,2]^2$
- $B = \{(x,y) \in \mathbb{R}^2 : \|(x,y) - (a,1)\| < 1\}$
- $A \setminus C = \emptyset$
- $B \setminus C = \emptyset$

a. **(1 pt)** State and justify the set of values of a such that $b \in A$ is a necessary condition for $b \in B$.

b. **(1.5 pts)** Define C and the set of values of a such that $\{A, B\}$ is a partition of C.

c. **(1.5 pts)** Consider the set, in $\mathbb{R}^3$, $D = (A \cup B) \times \{0\}$. Define the interior and the boundary of D.

4. **(5 pts)** State and justify the truth value of **five** of the following six propositions:

a. **(1 pt)** For every proposition $\mathcal{A}$, there is a proposition $\mathcal{B}$ such that $\mathcal{A} \vee \mathcal{B}$ is false.

b. **(1 pt)** If A and B are two sets, then $A \times B$ and $B \times A$ are disjoint.

c. **(1 pt)** If $a \in (\mathbb{R} \setminus \mathbb{Q})$ and $A = \{r \in \mathbb{R}^+ : B_r(a) \cap \mathbb{Q} \neq \emptyset\}$, then the infimum of A is 0.

d. **(1 pt)** If A is a compact set and $B \subset A$, then B is compact.

e. **(1 pt)** The several level sets of a real function form a partition of its domain.

f. **(1 pt)** If $f: A \to B$ is an invertible function and f^{-1} is its inverse, then $(f^{-1} \circ f)$ and $(f \circ f^{-1})$ are identical functions.

Solution Topics

1.

a.

- State that $\frac{|2x+1|-1}{x^2+|x|+1} \leq 0 \Leftrightarrow |2x+1|-1 \leq 0 \Leftrightarrow x \in [-1,0]$. Then, $A = [-1,0]$

b.

- State that $B =]-\infty, -1[$
- State that the supremum of B is -1, it is the least element that is greater than or equal to all elements of B
- Since the supremum of B do not belong to B, there is no maximum

c.

- State that $C =]-\infty, -1[\cup (]-1,0[\cap \mathbb{R}\backslash\mathbb{Q}) \cup]0, +\infty[$
- State that $int(C) =]-\infty, -1[\cup]0, +\infty[$
- State that $front(C) = [-1,0]$
- State that $C' = \mathbb{R}$

d.

- State that $C^c = [-1,0] \cap \mathbb{Q}$
- Conclude that C^c is bounded because there is at least an open ball that contains all elements of C, for example $B_2(0)$

e.

- State that C is not convex, because it is possible to find two different points of C whose line segment that connects them is not completely contained in C. For example, the line segment that connects -2 and 1 will include irrational numbers that do not belong to C (between -1 and 0)
- State that C is not connected. If $C_1 = A^c$ and $C_2 =]-1,0[\cap \mathbb{R}\backslash\mathbb{Q}$, then $C_1 \cap C_2 = \emptyset$, $C_1 \cap C_2 = C$, and $\overline{C_1} \cap C_2 = C_1 \cap \overline{C_2} = \emptyset$

  Alternatively, show first that C is not connected, and then state that is if C is not connected, C is not convex (sufficient condition)
- Conclude that the disjunction is false because both propositions are false

2.

a.

- Show that $D_f = \{(x,y) \in \mathbb{R}^2 : 2y - x > 0 \wedge x - y \geq 0\} = \left\{(x,y) \in \mathbb{R}^2 : \frac{x}{2} < y \leq x\right\}$.
- Geometrically represent D_f, the region below or on the line $y = x$, and above the line $y = \frac{x}{2}$, for $x > 0$.

b.

- Find the level set $L_{f_1}^k = \{(x,y) \in D_f \ : \ \log_2(2y - x) = k\ ,\ k \in \mathbb{R}\} =$

$$= \{(x,y) \in D_f \ : y = 2^{k-1} + \frac{x}{2}, k \in \mathbb{R}\}.$$

- Represent the level set $L_{f_1}^1 = \left\{(x,y) \in D_f \ : y = 1 + \frac{x}{2}\right\}$, bearing in mind D_f.
- Substitute the point $(3,2)$ in the general level curve of f_1 to find $k = 0$. Represent, bearing in mind D_f, $L_{f_1}^0 = \left\{(x,y) \in D_f \ : y = \frac{x+1}{2}\right\}$.

c.

- State that if f is invertible, then $\begin{cases} u = \log_2(2y - x) \\ v = \sqrt{x - y} \end{cases} \Leftrightarrow \begin{cases} 2^u = 2y - x \\ v^2 = x - y \end{cases} \Leftrightarrow \begin{cases} 2^u + v^2 = y \\ 2^u + 2v^2 = x \end{cases}$.
- Characterize the inverse function: $f^{-1} : \ R_f \to D_f\ ,\ (u,v) \longmapsto (2^u + 2v^2, 2^u + v^2)$.

d.

- State that the dimensions of the codomain of g and the domain of f are not compatible, so the composition $(f \circ g)$ does not exist.
- State that the dimensions of the codomain of f and the domain of g are compatible, so the composition $(g \circ f)$ may exist.
- Find the domain of $(g \circ f)$, $D_{g \circ f} = \left\{(x,y) \in \mathbb{R}^2 : (x,y) \in D_f \wedge f(x,y) \in D_g\right\}$
  - $D_g = \mathbb{R}^2$, so $\left[(x,y) \in D_f \Rightarrow f(x,y) \in D_g\right]$.
  - Conclude that $D_{g \circ f} = D_f$
- Find the expression of the inverse: $(g \circ f) = g\big(f(x)\big) = g\big(log_2(2y - x)\,, \sqrt{x - y}\big) = \frac{(x-y)^2+1}{2y-x}$.
- Characterize the inverse function: $(g \circ f) : D_f \to \mathbb{R}^2\ ,\ (x,y) \longmapsto \frac{(x-y)^2+1}{2y-x}$

e.

- State that h is odd means that for all $z \in D_h, (-z) \in D_h$ and $h(-z) = -h(z)$.
- State that $h(-z) = -h(z) \ \Rightarrow \ h^2(-z) = \big(-h(z)\big)^2 = h^2(z) \ \Rightarrow \ h^2$ is even.
- State that there are different inputs generating the same output, e.g. $i(1,1,1) = i(1,1,-1)$.
- State that if i is not one-to-one, then i is not invertible.

3.

a.

- State that if $b \in B$ is a necessary condition for $b \in A$ (or equivalently, $b \in A$ is a sufficient condition for $b \in B$), then $B \subset A$
- State and justify that a must be equal to 1, so that $B \subset A$

b.

- State that if $\{A, B\}$ is a partition of C, then both $A \cap B = \emptyset$ and $A \cup B = C$ must be true.
- Conclude that we must have $(a \leq -1 \vee a \geq 3)$ so that A and B are disjoint sets, and $C = A \cup B$ (C cannot have elements not belonging to $A \cup B$).

c.

- Describe D or, alternatively, represent D in $\mathbb{R}^3$
- Justify that $\operatorname{int}(D) = \emptyset$
- Justify that $\operatorname{fr}(D) = (A \cup \bar{B}) \times \{0\}$

4.

a. State that the proposition is false because, for example, if $\mathcal{A}$ is "Sporting Clube de Portugal was the champion in adult male football in 2020 – 2021.", a true proposition, then $\mathcal{A} \vee \mathcal{B}$ is true regardless of the truth value of $\mathcal{B}$.

b. State that the proposition is false because, for example, if $A = \{0,1\}$ and $B = \{1,2\}$, then $(1,1) \in (A \times B) \cap (B \times A)$.

c. State that the proposition is true because $\forall r \in \mathbb{R}^+, B_r(a) \cap \mathbb{Q} \neq \emptyset$, which means that $A = \mathbb{R}^+$.

d. State that the proposition is false because, for example, if $A = [-2,2]^2$ and $B = B_1(0,0)$, then A is compact, $B \subset A$ and B is not compact.

e. State that the proposition is true because each point in the domain of a real function has a single image, which means it belongs to exactly one level set of that function and so the intersection of any two level sets is empty and the union of all the level sets is the domain.

f. State that the proposition is false because, for example, if $f: \mathbb{R} \to \mathbb{R}^+$ is defined by $f(x) = e^x$, then $f^{-1}: \mathbb{R}^+ \to \mathbb{R}$ is defined by $f^{-1}(y) = \ln y$ and $D_{f^{-1} \circ f} = \mathbb{R} \neq \mathbb{R}^+ = D_{f \circ f^{-1}}$.