**Nova School of Business and Economics**
**Calculus I 2021 – 2022 S1**
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NOVA SCHOOL OF BUSINESS & ECONOMICS

# Resit Exam

- **Date:** January 25, 2022
- **Duration:** 2 hours and 30 minutes
- **Instructions: 1:** The exam has **five questions. 2:** Write your number and **absolutely nothing else** on this exam paper, and **hand it in at the end. 3:** Write your answers on the answer booklet, using the **front and back** of each sheet, **stating** the question you are answering, **never** answering **more than one question on the same sheet**, and **not unstapling** any sheets. **4:** If you want to use any sheet of the answer booklet as space for **drafts**, state it clearly on the **space for the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7:** If, in question **5**, you answer all parts, **only the first four you answer will be graded**. **8: No** individual questions about the exam will be answered. **9:** Break a leg (not literally)!

| **Nº:** |
| --- |

**1.** **(3.5 pts)** Consider the functions $f: \mathbb{R}^2 \to \mathbb{R}^2$ and $g: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$, defined, respectively, by:

$$f(x,y) = (2x + y, x - 2y) \qquad\qquad g(u,v) = \frac{(u-v)^2}{u^2 + v^2}$$

**a.** **(1 pts)** Knowing that $f$ is invertible, characterize the inverse of $f$.

**b.** **(1.75 pts)** Consider the function $h = g \circ f$.

**(i)** **(0.75 pts)** Show that $h$ is defined by $h(x,y) = \frac{(x+3y)^2}{5(x^2+y^2)}$.

**(ii)** **(1 pts)** State and justify whether there is a continuous extension of $h$ to $(0,0)$.

**c.** **(0.75 pts)** Consider a function $i: D \subset \mathbb{R}^2 \to \mathbb{R}$, such that $i'_x = h$ and $i'_y(1,0) \in \mathbb{R}$. State and justify whether $i$ is differentiable at $(1,0)$.

**2.** **(3.5 pts)** Consider the function $f: D \subset \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \frac{2 + x + ln\,(3 - x^2 - x)}{2x + 4}$$

**a.** **(1 pt)** State and justify whether there is a continuous extension of $f$ to $-2$.

**b.** **(1 pt)** Show that the graph of $f$ intersects the curve defined by $y = 2x^2$ at at least two points whose $x -$ values are, respectively, in $[-1,0]$ and $[0,1]$.

**c.** **(1.5 pts)** Consider the function $g: D \subset \mathbb{R} \to \mathbb{R}$, defined by $g(x) = f(x) - 2x^2$. Without further computations:

**(i)** **(0.75 pts)** State and justify whether $g$ is invertible.

**(ii)** **(0.75 pts)** Show that the minimum of $|g'|$ is $0$.

**3.** **(4.5 pts)** Consider the set $A = \left\{(x,y) \in \mathbb{R}^2 : \left(|y| - \frac{1}{x}\right) x \leq 0\right\}$ and the function $f: \mathbb{R}^2 \to \mathbb{R}$, defined by:

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{\sqrt{4+|x|+|y|}} & \text{if} \quad (x,y) \in A \\ xy & \text{if} \quad (x,y) \notin A \end{cases}$$

**a.** **(1 pt)** State and justify whether $A$ is connected.

**b.** **(2 pts)** State and justify whether $f$ is differentiable at $(0,0)$.

**c.** **(1.5 pts)** Consider the function $g: \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^+$, defined by $g(u,v) = e^{f'_{(u,v)}(2,2)}$. Define the level $1$ set of $g$.

**Observation:** $(2,2) \in \text{ext}(A)$.

**4.** **(4.5 pts)** Consider the sequences $(u_n)$ and $(v_n)$, defined, respectively, by $u_n = \sum_{i=1}^{100}\left(i + \frac{i}{n}\right)$ and $v_n = 1 + \frac{1}{n}$.

**a.** **(1.25 pt)** Show that $u_n = 5050 \cdot v_n$.

**b.** **(1 pt)** Compute $\lim u_n$, and confirm your answer using the definition of limit of a sequence.

**c.** **(2.25 pts)** Consider $k = \lim \left[v_n^{\left(\frac{1}{n} - 2n\right)}\right]$.

**(i)** **(1.25 pts)** Show that $k = \frac{1}{e^2}$.

**(ii)** **(1 pts)** State and justify whether $\sum_{n=1}^{+\infty} \sqrt{k^n}$ is convergent and, if so, compute its sum.

**5.** **(4 pts)** State and justify the truth value of **four** of the following five propositions:

**a.** **(1 pt)** If $A$ is a convex set and $\mathcal{P}$ is a partition of $A$, then $\exists B \in \mathcal{P}: B$ is convex.

**b.** **(1 pt)** If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^m \to \mathbb{R}^p$ are two functions such that $g \circ f$ exists, then the range of $g \circ f$ is a subset of the range of $g$.

**c.** **(1 pt)** If $A \subset \mathbb{R}^n$, $a \in \text{fr}(A)$ and $f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^n \to \mathbb{R}^m$ and $h: \mathbb{R}^n \to \mathbb{R}^m$ are three functions such that $g$ and $h$ are continuous at $a$ and $f$ is defined by $f(x) = \begin{cases} g(x) & \text{if} \quad x \in A \\ h(x) & \text{if} \quad x \notin A \end{cases}$ then $f$ is continuous at $a$.

**d.** **(1 pt)** If $a \in \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}$ is a function differentiable at $a$, then the largest directional derivative of $f$, along a vector with the norm of $\nabla_f(a)$, at $a$, is $\left\|\nabla_f(a)\right\|^2$.

**e.** **(1 pt)** If $u_1 \in \mathbb{R} \setminus \{0\}$, $r \in ]0,1[$ and $(u_n)$ is a geometric sequence with first term $u_1$ and common ratio $r$, then $\sum_{n=1}^{+\infty} u_n > \sum_{n=2}^{+\infty} u_n$.

# Solution Topics

**1.**

**a.**

- Solve the system of simultaneous equations $\begin{cases} s = 2x + y \\ t = x - 2y \end{cases} \Leftrightarrow \begin{cases} x = \frac{2s+t}{5} \\ y = \frac{s-2t}{5} \end{cases}$
- Characterize the inverse function: $f^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$ , $(s,t) \to \left(\frac{2s+t}{5}, \frac{s-2t}{5}\right)$

**b.**

**(i)**

- State that $h(x,y) = g\big(f(x,y)\big) = g(2x+y, x-2y)$
- Substitute $u = (2x+y)$ and $v = (x-2y)$ in the expression of $g$ and simplify correctly.

**(ii)**

- Find two limits along different trajectories that are different, for example, $\lim\limits_{\substack{(x,y)\to(0,0) \\ y=0}} h(x,y) = \frac{1}{5}$ and $\lim\limits_{\substack{(x,y)\to(0,0) \\ x=0}} h(x,y) = \frac{9}{5}$
- Conclude that $\nexists \lim\limits_{(x,y)\to(0,0)} h(x,y)$ , we cannot define a continuous extension of $h$ to $(0,0)$.

**c.**

- Justify that $h$ is differentiable at $(1,0)$ since it is the result of operations with differentiable functions. Therefore, $i'_x$ is also differentiable at $(1,0)$.
- State that since $i'_x$ is differentiable at $(1,0)$, it is also continuous at $(1,0)$.
- State that since $i'_x$ and $i'_y$ both exist at $(1,0)$, and $i'_x$ is continuous at $(1,0)$, then $h$ is differentiable at $(1,0)$.

**2.**

**a.**

- State that we can only define a continuous extension of $f$ to $x = -2$ if the limit as $x \to -2$ exists and is finite.
- Identify the indeterminate form $\left(\frac{0}{0}\right)$, and verify that $\lim_{x \to -2} f(x) = 2$, using the Cauchy rule.
- Conclude that we can define a continuous extension of $f$ to $x = -2$

**b.**

- Define a function $g$ such that $g(x) = f(x) - 2x^2 = \frac{2+x+\ln(3-x^2-x)}{2x+4} - 2x^2$.
- Justify that $g$ is a continuous function in all its domain, since it is the result of operations between continuous functions. Therefore, $g$ is also continuous on $[-1,0]$ and on $[0,1]$.
- Compute $g(-1), g(0), g(1)$ and verify that $g(-1) \times g(0) < 0$ and that $g(0) \times g(1) < 0$.
- State that the Corollary to the Intermediate Value Theorem (Bolzano's Theorem) ensures that in each of the two given intervals, there is a zero of $g$, which means that $f(x) = 2x^2$.

**c.**

**(i)**

- State that as a consequence of what is mentioned in part b), function $g$ has two zeros, that is, two values for $x$ with the same image $y = 0$. Which means that $g$ is not injective (or one-to-one).
- State that since $g$ is not injective, it is not invertible.

**(ii)**

- Justify that $g$ is a differentiable function, since it is the result of operations involving differential functions.
- Letting $c_1$ and $c_2$ be the zeros of $g$ whose existence was shown before, state that $g$ is a continuous function on $[c_1, c_2]$ and a differentiable function on $]c_1, c_2[$ (and, of course, $f(c_1) = f(c_2) = 0$).
- State that given that its conditions are satisfied, Rolle's Theorem ensures the existence of a point $p \in ]c_1, c_2[$ such that $g'(p) = 0$.
- State that given that $|g'(x)| \geq 0$ and $\exists p \in D_g: g'(p) = 0$, then the minimum of $|g'|$ is zero.

**3.**

**a.**

- Solve the condition $\left(|y|-\frac{1}{x}\right)x \leq 0$.
- Conclude that $A$ is a disconnected set, presenting two subsets of $A$, $A_1$ and $A_2$, such that: $A_1 \cap \overline{A_2} = \emptyset$, $\overline{A_2} \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$
  For example: $A_1 = \left\{(x,y) \in \mathbb{R}^2 : y \leq \frac{1}{x} \wedge y \geq \frac{1}{x} \wedge x > 0\right\}$ and $A_2 = \{(x,y) \in \mathbb{R}^2 : x < 0\}$

**b.**

- Compute $f'_y(0,0) = 0$, using the differentiation rules or the definition of partial derivative of $f$ with respect to $y$.
- Compute $f'_x(0,0) = 0$, using the definition of partial derivative of $f$ with respect to $x$.
- Find the tangent plane to the graph of $f$ in the origin: $z = 0$.
- Write $\lim\limits_{\substack{(x,y)\to(0,0)\\(x,y)\notin A}} \frac{f(x,y)-0}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.
- Prove, using the definition of limit, that $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$.
- Compute $\lim\limits_{\substack{(x,y)\to(0,0)\\(x,y)\in A}} \frac{f(x,y)-0}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{4-|x|-|y|}\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{\sqrt{4-|x|-|y|}} = 0$.
- Conclude that $f$ is differentiable at $(0,0)$.

**c.**

- Write $g(u,v) = 1 \Leftrightarrow e^{f'_{(u,v)}(2,2)} = 1 \Leftrightarrow f'_{(u,v)}(2,2) = 0$.
- State that $f$ is differentiable at $(\mathbf{2},\mathbf{2})$ since this is an interior point of $A^c$, where the function is defined by a polynomial.

**Option 1:**

- State that $f$ is differentiable at $(\mathbf{2},\mathbf{2})$, therefore $f'_{(u,v)}(2,2) = 0 \Leftrightarrow (u,v) \perp \nabla_f(2,2)$.
- Compute $\nabla_f(2,2) = \left(f'_x(2,2), f'_y(2,2)\right) = (y,x)_{(2,2)} = (2,2)$.
- Conclude that $(u,v) = k(-2,2) = (-2k, 2k)$, where $k \in \mathbb{R}$.

**Option 2:**

- State that $f$ is differentiable at $(2,2)$, therefore $f'_{(u,v)}(2,2) = 0 \Leftrightarrow f'_x(2,2) \times u + f'_y(2,2) \times v = 0$.
- Compute $f'_x(2,2)$ and $f'_y(2,2)$.
- Find the condition $2u + 2v = 0$ and conclude that $u = -v$.
- Conclude that $(u,v) = k(-2,2) = (-2k, 2k)$, where $k \in \mathbb{R}$.

**4.**

**a.**

- Show that $u_n = 5050 \cdot v_n$, using the following facts:
- $\sum_{i=1}^{100}\left(i + \frac{i}{n}\right) = (1 + 2 + \cdots + 100) + \left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{100}{n}\right) = (1 + 2 + \cdots + 100) + \frac{1}{n}(1 + 2 + \cdots + 100) = \left(1 + \frac{1}{n}\right)(1 + 2 + \cdots + 100)$
- $1 + 2 + \cdots + 100$ is the sum of the first terms of an arithmetic sequence $(r = 1, u_1 = 1)$, $S_{100} = \frac{1+100}{2} \times 100 = 5050$

**b.**

- Compute $\lim u_n$.
- Prove, using the definition, that $\lim u_n = 5050$.

**c.**

**(i)**

- Write $\lim\left[v_n^{\left(\frac{1}{n}-2n\right)}\right] = \frac{\lim \sqrt[n]{v_n}}{\lim(v_n)^{2n}}$.
- Compute $\lim \sqrt[n]{v_n} = \lim \sqrt[n]{1 + \frac{1}{n}}$, using, for example, the following result: $\lim \frac{1+\frac{1}{n+1}}{1+\frac{1}{n}} \to a \Rightarrow \lim \sqrt[n]{1 + \frac{1}{n}} \to a$ , given that $1 + \frac{1}{n} \geq 0, \forall n \in \mathbb{N}$.
- Write that $\lim(v_n)^{2n} = \lim\left(1 + \frac{2}{2n}\right)^{2n} = e^2$.
- Conclude that $k = \lim\left[v_n^{\left(\frac{1}{n}-2n\right)}\right] = \frac{1}{e^2}$.

**(ii)**

- Write $\sum_{n=1}^{+\infty}\sqrt{\left(\frac{1}{e^2}\right)^n} = \sum_{n=1}^{+\infty}\left(\frac{1}{e}\right)^n$
- Classify $\sum_{n=1}^{+\infty}\left(\frac{1}{e}\right)^n$ as a geometric series, since $z_n = \left(\frac{1}{e}\right)^n$ is a geometric sequence.
- Conclude that $\left|\frac{1}{e}\right| < 1$, therefore the geometric series is convergent.
- Conclude that the sum of the series is $S = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}$.

**5.**

**a.** State that the proposition is false because, for example, if $A=[-1,1]^2$, $B=\{(x,y)\in[-1,1]^2: y\leq x^3\}$, $C=\{(x,y)\in[-1,1]^2: y>x^3\}$ and $\mathcal{P}=\{B,C\}$, then $A$ is convex, $\mathcal{P}$ is a partition of $A$ and $B$ and $C$ are not convex

**b.** State that the proposition is true because all images of $g\circ f$ are images of $g$

**c.** State that the proposition is false because, for example, if $g$, $h$ and $f$ are defined, respectively, by $g(x)=e^x$, $h(x)=x$ and $f(x)=\begin{cases} g(x) & \text{if} \quad x<0 \\ h(x) & \text{if} \quad x\geq 0\end{cases}$, then $g$ and $h$ are continuous at $0$, but $f$ is not.

**d.** State that the proposition is true because, as $f$ is differentiable at $a$, the largest directional derivative of $f$, along a vector with the norm of $\nabla_f(a)$, at $a$, is $f'_{\nabla_f(a)}(a)=\nabla_f(a)\cdot\nabla_f(a)$

**e.** State that the proposition is false because, for example, if $(u_n)$ is defined by $u_n=-\frac{1}{2^n}$, then $\sum_{n=1}^{+\infty}(u_n)=-1<-\frac{1}{2}=\sum_{n=2}^{+\infty}(u_n)$