Nova School of Business and Economics
Calculus I 2021 – 2022 S1
Carolina Nogueira **Inês Legatheaux** **Pamela Pacciani**
Catarina Ângelo **Jessica Lomba** **Pedro Chaves**
Francisco Rodrigues **João Farinha** **Salvador Murteira**
Helena Almeida

Regular Exam

- **Date:** January 5, 2022
- **Duration:** 2 hours and 30 minutes
- **Instructions: 1:** The exam has **five questions. 2:** Write your number and **absolutely nothing else** on this exam paper, and **hand it in at the end. 3:** Write your answers on the answer booklet, using the **front and back** of each sheet, **stating** the question you are answering, **never** answering **more than one question on the same sheet**, and **not unstapling** any sheets. **4:** If you want to use any sheet of the answer booklet as space for **drafts**, state it clearly on the **space for the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7:** If, in question **5**, you answer all parts, **only the first four you answer will be graded. 8: No** individual questions about the exam will be answered. **9:** Break a leg (not literally)!

Nº:

1. **(4.5 pts)** Consider the function $f: D_f \subset \mathbb{R}^2 \to \mathbb{R}^2$, defined by:

$$f(x,y) = \big(f_1(x,y), f_2(x,y)\big) = \left(y - (x^2 - x)^2, x\right)$$

a. **(1 pt)** Define D_f ~~and the range of f~~.

b. **(1 pt)** State and justify whether f is invertible and, if so, characterize its inverse.

c. **(1.5 pts)** Consider the function $g: \mathbb{R} \to \mathbb{R}$, whose graph is the $-\frac{1}{32}$ level set of f_1. Without computing it, show that g has exactly one zero in $\left]\frac{1}{2}, 1\right[$.

d. **(1 pt)** Consider $a \in \mathbb{R}^+$, the functions $h: D_h \subset \mathbb{R}^2 \to \mathbb{R}$, defined by $h(u,v) = \frac{1}{|v|-1}$ and $i: D_i \subset \mathbb{R}^2 \to \mathbb{R}$, such that $i = h \circ f$, and $A = D_i \cup]-a, a]^2$. Define the set of values of a such that A is path – connected.

2. **(4 pts)** Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$, defined by:

$$f(x,y) = \begin{cases} \dfrac{x^2y + xy^3}{x^2 + y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

a. **(1 pt)** Define the continuity domain of f.

b. **(1 pt)** Compute $\nabla_f(0,0)$.

c. **(1 pt)** Compute $f'_{(2,2)}(0,0)$.

d. **(1 pt)** Based on your answers to **b.** and **c.**, state and justify whether f is differentiable at $(0,0)$.

3. **(3.5 pts)** Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$, defined by:

$$f(x,y) = \begin{cases} y^2 - x^2 & \text{if} \quad y > 0 \\ 0 & \text{if} \quad y \leq 0 \end{cases}$$

a. **(1.5 pts)** Define the differentiability domain of f.

b. **(2 pts)** Consider $A = \bar{B}_1(0,0) \setminus B_1(0,0)$ and the function $g: \mathbb{R}^2 \to \mathbb{R}$, defined by $g(u,v) = f'_{(u,v)}(1,2)$.

(i) **(1 pt)** Without computing it, show that g has a global minimum in A.

(ii) **(1 pt)** Compute the global minimum whose existence you proved in **(i)**.

4. **(4 pts)** Consider the sequence (u_n), defined by:

$$u_n = \begin{cases} [1 + (-1)^{n+1}] \cdot n & \text{if} \quad n \leq 1000 \\ 2^{998-n} & \text{if} \quad n > 1000 \end{cases}$$

a. **(0.5 pts)** Compute u_1, u_2, u_{1000} and u_{1001}.

b. **(1 pt)** Compute $\lim u_n$, and confirm your answer using the definition of limit of a sequence.

c. **(0.75 pts)** Consider $A = \{x \in \mathbb{R}: x = u_n \wedge n \in \mathbb{N}\}$. Define the boundary, the closure and the derived set of A.

d. **(1 pt)** Consider (S_n), the sequence of partial sums of (u_n). Compute S_4 and S_{1000}.

e. **(0.75 pts)** State and justify whether $\sum_{n=1}^{+\infty} u_n$ is convergent and, if so, compute its sum.

5. **(4 pts)** State and justify the truth value of **four** of the following five propositions:

a. **(1 pt)** If A and B are two disconnected sets, then $A \cap B$ is disconnected.

b. **(1 pt)** If $\#(S)$ represents the cardinal number of set S, A and B are sets, $\#(A) \in \mathbb{N}$ and $f: A \to B$ is a function with range C, then $\#(C) \leq \#(A)$.

c. **(1 pt)** If $f: \mathbb{R}^n \to \mathbb{R}^m$ is an injective function and the range of f_1 is C_1, then, for every $a \in C_1$, there is a unique $x \in \mathbb{R}^n$ such that $f_1(x) = a$.

d. **(1 pt)** If $a, b \in \mathbb{R}$, $a < b$ and $f: [a,b] \to \mathbb{R}$ is a non − constant and continuous function, then there are $c, d \in \mathbb{R}$ such that the range of f is $[c,d]$.

e. **(1 pt)** If (u_n) is a sequence with limit 1 and (v_n) is the sequence defined by $v_n = \min\{0, u_n\}$, then $\sum_{n=1}^{+\infty}(v_n)$ is convergent.

Solution Topics

1.

a.

- State that $D_f = \mathbb{R}^2$, since there are no restrictions to the values of the variables x and y.
- State that $R_f = \mathbb{R}^2$, since the two entries of the outputs of f are independent. For each value of x, y can be any real number, so the outputs of f can be any element of $\mathbb{R}^2$.

b.

- State that f is invertible if it is bijective (simultaneously invertible (on-to-one) and surjective (onto)).
- Prove that f is injective, there are no two inputs with the same output:
  $f(x_1, y_1) = f(x_2, y_2) \Rightarrow (x_1, y_1) = (x_2, y_2)$
- Prove that f is surjective. Since $R_f = \mathbb{R}^2$ and $CD_f = \mathbb{R}^2$, then $R_f = CD_f$.
- Conclude that f is invertible.
- Characterize f^{-1}, stating its domain ($\mathbb{R}^2$), codomain ($\mathbb{R}^2$) and its analytical expression $f^{-1}(u, v) = (v, u + (v^2 - v)^2$.

c.

- Find $g(x) = (x^2 - x)^2 - \frac{1}{32}$
- Justify that g is continuous in $\mathbb{R}$ since it is a polynomial function, so it is continuous in $\left[\frac{1}{2}, 1\right]$.
- State that $g\left(\frac{1}{2}\right) \times g(1) < 0$ and conclude, using the intermediate value theorem (Bolzano-Cauchy theorem), that g has at least one zero in $\left]\frac{1}{2}, 1\right[$.
- Solve $g'(x) = 0 \Leftrightarrow x = 0 \vee x = \frac{1}{2} \vee x = 1$.
- Conclude that g' is continuous in $\left[\frac{1}{2}, 1\right]$ and differentiable in $\left]\frac{1}{2}, 1\right[$ since it is a polynomial function.
- If $\frac{1}{2}$ and 1 are two consecutive zeros of g' then, using the second corollary of the Rolle´s Theorem, g has a maximum of one zero in $\left]\frac{1}{2}, 1\right[$.
- Conclude that g has exactly one zero in $\left]\frac{1}{2}, 1\right[$.

d.

- Show that $i(x, y) = (h \circ f)(x, y) = \frac{1}{|x|-1}$.
- State that $D_i = (\mathbb{R} \setminus \{-1, 1\}) \times \mathbb{R}$.
- Conclude that $a > 1$ so that A is a path-connected set.

2.

a.

- Explain why f is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$
- State that $f(0,0) = 0$
- Show, using the definition of limit of a function, and finding an example of a distance in the inputs for each distance in the outputs δ, such as $\min\left\{1, \frac{\delta}{2}\right\}$, that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$
- Explain why f is continuous at $(0,0)$
- Show that the continuity domain of f is $\mathbb{R}^2$

b.

- Show, using the definition of partial derivative, that $f'_x(0,0) = 0$
- Show, using the definition of partial derivative, that $f'_y(0,0) = 0$
- State that $\nabla_f(0,0) = (0,0)$

c. Show, using the definition of directional derivative, that $f'_{(2,2)}(0,0) = 1$

d.

- State that, if f is differentiable at $(0,0)$, then $f'_{(2,2)}(0,0) = 2f'_x(0,0) + 2f'_y(0,0)$
- Show that $f'_{(2,2)}(0,0) \neq 2f'_x(0,0) + 2f'_y(0,0)$, which means that f is not differentiable at $(0,0)$

3.

a.

- State that $D_f = \mathbb{R}^2$.
- Justify that f is differentiable for $(x, y) \in \mathbb{R}^2 : y > 0$ since f is a polynomial function.
- Justify that f is differentiable for $(x, y) \in \mathbb{R}^2 : y < 0$ since f is a polynomial function (degree 0).
- Study the differentiability of f for $(x, y) \in \mathbb{R}^2 : y = 0$, points that can be written as $(a, 0), a \in \mathbb{R}$:
  - Prove that $f'_x(a, 0) = 0$ using the definition of partial derivative of a function or the differentiation rules.
  - Prove that $f'_y(a, 0^-) = 0$ using the definition of partial derivative of a function or the differentiation rules.
  - Compute $f'_y(a, 0^+)$ using the definition of partial derivative and conclude that it will be 0 if $a = 0$ and it will not exist if $a \neq 0$.
  - Conclude that f cannot be differentiable at $(a, 0), a \neq 0$, since $\nexists f'_y(a, 0^+)$.
  - Write the equation of the plane tangent to the graph of f at $(0,0)$:
    $$z = f(0,0) + f'_x(0,0)(x - 0) + f'_y(0,0)(y - 0) = 0$$
  - Show that $\lim_{(x,y)\to(0,0)} \frac{f(x,y)-0}{\|(x,y)\|} = 0$:
    - $\lim_{\substack{(x,y)\to(0,0)\\ y\leq 0}} \frac{f(x,y)-0}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{0}{\|(x,y)\|} = 0$
    - $\lim_{\substack{(x,y)\to(0,0)\\ y>0}} \frac{f(x,y)-0}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{\|(x,y)\|} = \frac{0}{0}$ (ind)
    - Using the definition of limit, show that the limit is 0, for example using the relation $\varepsilon = \delta$.
  - Conclude that f is differentiable at $(0,0)$ and for $(x, y), y \neq 0$. Therefore, the differentiability domain of f is $\mathbb{R}^2 \backslash \{(a, 0), a \neq 0\}$.

b.

(i)

- State that f is differentiable at $(1,2)$, following the previous question.
- State that if f is differentiable at $(1,2)$, then $f'_{(u,v)}(1,2) = f'_x(1,2) \times u + f'_y(1,2) \times v = -2u + 4v$
- Justify that g is continuous on $D_g = \mathbb{R}^2$ since it is a polynomial function, hence continuous on $A \subset D_g$.
- Justify that A is compact since it is both closed and bounded:
  - A is closed since $A = \bar{A}$
  - A is bounded since $A \subset B_2(0,0)$
- Conclude that, following the Weierstrass extreme-value theorem, g must have a global minimum on A.

(ii)

- Conclude that the minimum of $g(u, v) = f'_{(u,v)}(1,2)$ on A is attained for $(u, v) = -\frac{\nabla_f(1,2)}{\|\nabla_f(1,2)\|} = -\frac{(-2,4)}{\sqrt{20}} = \left(\frac{2}{\sqrt{20}}, -\frac{4}{\sqrt{20}}\right)$. The minimum is, then, $f'_{\left(\frac{2}{\sqrt{20}}, -\frac{4}{\sqrt{20}}\right)}(1,2) = -2 \times \frac{2}{\sqrt{20}} + 4 \times \left(-\frac{4}{\sqrt{20}}\right) = -\sqrt{20}.$

4.

a.

- Compute $u_1 = 2$, $u_2 = 0$, $u_{1000} = 0$ using the upper branch of the sequence.
- Compute $u_{1001} = \frac{1}{8}$ using the lower branch of the sequence.

b.

- Compute $\lim u_n = \lim 2^{998-n} = \lim \left(\frac{1}{2}\right)^{n-998} = 0$.
- Show that for all $\delta > 0$, if we let p be the first natural index larger than $998 - \log_2 \delta$, we have $(n \geq p) \Rightarrow |u_n - 0| < \delta$, which proves that $\lim u_n = 0$.

c.

- $front(A) = \partial A = A = \bar{A}$
- $A' = \{0\}$

d.

- $S_4 = u_1 + u_2 + u_3 + u_4 = 2 + 0 + 6 + 0 = 8$
- Realize that when n is even $n \leq 1000$, (u_n) is always zero.
- Realize that for n odd and $n \leq 1000$, we have the first 500 terms of na arithmetic sequence, with common difference $d = 4$.
- $w_n = u_{2n-1} = 2(2n-1)\,, n \leq 500$, or $w_n = \{2,6,10, \dots, 2 \times 999\}, n \leq 500$.
- $S_{1000} = 2 + 0 + 6 + 0 + 8 + \cdots + 1998 + 0 = 2 + 6 + \cdots 1994 + 1998 = \frac{1998+2}{2} \times 500 = 1000 \times 500 = 500{,}000$, where we used the formula of the sum of the first n terms of an arithmetic sequence.

e.

- Realize that starting at $n = 1001$, we have a geometric sequence, with common ratio $r = \left(\frac{1}{2}\right)$.
- State that ince $|r| < 1$, the series $\sum_{n=1001}^{\infty} u_n = \sum_{n=1001}^{+\infty} \left(\frac{1}{2}\right)^{n-998}$ converges, and its sum is given by the expression $u_{1001} \times \frac{1}{1-\frac{1}{2}} = \frac{1}{8} \times 2 = \frac{1}{4}$.
- Write $\sum_{n=1}^{+\infty} u_n = S_{1000} + \sum_{n=1001}^{\infty} \left(\frac{1}{2}\right)^{n-998} = 500{,}000 + \frac{1}{4} = 500{,}000.25$

5.

a. State that the proposition is false because, for example, if $A = B_1(0,0) \cup [2,4] \times [-1,1]$ and $B = B_1(0,0) \cup \{(2,2)\}$, then A and B are disconnected and $A \cap B$ is connected

b. State that the proposition is true because, if f has n inputs and, to each input of f corresponds a single output, then f has at most n outputs

c. State that the proposition is false because, for example, if f is defined by $f(x,y) = (x+y, x-y)$, then f is injective and $f_1(1,-1) = f_1(-1,1) = 0$

d. State that the proposition is true because, as f is not constant and continuous in $[a,b]$ and $[a,b]$ is compact, f has global minimum and maximum in $[a,b]$, c and d, respectively, with $c \neq d$, and, as f is continuous in $[a,b]$, all real numbers in $[c,d]$ are outputs of at least one input of f in $[a,b]$

e. State that the proposition is true because, starting from a certain index m, all term of (u_n) are, for example, in $\left]\frac{1}{2}, \frac{3}{2}\right[$, which means that, starting from the index m, $v_n = 0$ and, so, $\sum_{n=1}^{+\infty}(v_n) = \sum_{n=1}^{m-1}(v_n) \in \mathbb{R}$