Nova School of Business and Economics
2020 – 2021 S2
Calculus I
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Resit Exam

- **Date:** June 22, 2021
- **Duration:** 2 hours and 30 minutes
- **Instructions: 1:** This exam has **four questions**. **2:** Write your number and **absolutely nothing else** in this exam, and **hand it in at the end**. **3:** Answer the exam on the answer book, using the **front** and **back** of each sheet, **stating** which question you are answering to, **never** answering **more than one question in the same sheet**, and **not unstapling** any sheet. **4:** If you want to use any sheet of the answer book as space for **drafts**, state it **on the space for the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7: No** individual questions about the exam will be answered. **8:** Break a leg (not literally)!

Number:

1. **(4 pts)** State, and justify, the truth value of the following statements:

a. **(1pt)** Consider the function $f: D_f \to \mathbb{R}$ defined by $f(x) = \ln(x+1) + \sqrt{2-x^2}$. Then, the sets D_f and $D_f \cap \mathbb{Q}$ do not have a *minimum*, but they have a *maximum*.

b. **(1 pt)** The series $\sum_{n\geq 1} \frac{(x-1)^n}{2^n}$ is convergent for $-1 < x < 3$, and if $x = 2$, the sum of the series is equal to 1.

c. **(1 pt)** Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\nabla f(1\,,1) = (2\,,3)$ and $f'_{(4,1)}(1,1) = 5$. Then, we cannot conclude anything regarding the differentiability of f at $(1,1)$.

d. **(1 pt)** Let $g: \mathbb{R} \to \mathbb{R}$, and $a, b \in \mathbb{R}$. It is known that g is of class C^∞, and that the tangent line to the graph of g is horizontal only at $x = a$ and $x = b$. Then, $g(x) = 0$ cannot have two solutions on $]a, b[$, and $\exists c \in \,]a, b[: g''(x) = 0$.

2. **(6 pts)** Consider the sequences with the following general terms:

$$u_n = 1 + 2v_n, \text{ where } |v_n| < 4, \forall n \in \mathbb{N} \qquad\qquad w_n = \left(\frac{2n-1}{2n}\right)^{n+1}$$

a. **(1.25 pts)** Show that (u_n) is bounded and, using the Squeeze Theorem, find $\lim \frac{u_n}{n^5+1}$.

b. **(0.75 pts)** Let (S_n) be the sequence of the partial sums of (w_n). Show that $S_1 + S_2 > \frac{1}{2}$.

c. **(1 pt)** Study regarding convergence the series $\sum_{n\geq 1} w_n$. Justify.

d. **(2 pts)** Consider the set $S = \left\{\left(\sqrt[n]{w_n}, \ln w_n\right) \in \mathbb{R}^2 : n \in \mathbb{N}\right\}$. Find the $int(S)$ and S'. Is S a compact set? Justify.

e. **(1 pt)** Let $T = [a,b] \times [c,d]$, where $a,b,c,d \in \mathbb{R}$. Find and justify the truth value of the proposition: $\exists\, a,b,c,d \in \mathbb{R}: S \cup T$ is a connected set.

3. **(5 pts)** Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2} & , \ (x,y) \neq (0,0) \\ 0 & , \ (x,y) = (0,0) \end{cases}$.

a. **(1.5 pts)** Compute $f_x'(0,0)$ and $f_y'(0,0)$.

b. **(1.5 pts)** Study f regarding differentiability at the origin.

c. Consider the function g, defined by $g(x,y) = f(x-y, x^2-1)$.

(i) **(1 pt)** Compute $g_x'(1,1)$

(ii) **(1 pt)** Assuming $H_f(0,0) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, compute $g_{xy}''(1,1)$.

4. **(5 pts)** Let $g: D_g \subset \mathbb{R}^2 \to \mathbb{R}$ be defined by the expression $g(x,y) = \frac{(2x+4)^2 y^2}{\sqrt{(x+2)^2+y^2}}$, and $f: D_f \subset \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$f(x,y) = \begin{cases} g(x,y) & , (x,y) \notin [-3,-1] \times [-1,0] \\ \ln\left(\dfrac{x^2+y^2}{4}\right) & , (x,y) \in [-3,-1] \times [-1,0] \end{cases}$$

a. **(1.5 pts)** Study g regarding the existence of limit at $(-2,0)$.

b. **(1 pt)** Find D_f (justify), and geometrically represent the level curve with level 0 of f.

c. **(1.5 pts)** Study f regarding continuity at points of the form $(a,0)$, where $a \in \mathbb{R}$.

d. **(1 pt)** Find the equation of the plane tangent to the graph of f at $(-2,0,f(-2,0))$.

Solution Topics

1.

a.

- Obtain $D_f = \{x \in \mathbb{R}: x + 1 > 0 \wedge 2 - x^2 \geq 0\} = \left]-1, \sqrt{2}\right]$.
- Conclude that the $inf\, D_f = -1$ does not belong to D_f and that the set has no minimum.
- Conclude that the $sup\, D_f \cap \mathbb{Q} = \sqrt{2}$ does not belong to the set given that $\sqrt{2}$ is not a rational number.
- The statement is FALSE.

b.

- Conclude that we are given a geometric series whith ratio $\frac{x-1}{2}$.
- State that the series is convergent if and only if $\left|\frac{x-1}{2}\right| < 1$.
- Conclude the series is convergent for $-1 < x < 3$.
- Obtain the value 1 as sum of the series when $x = 2$.
- The statement is TRUE.

c.

- Mention that if f is differentiable at $(1,1)$ then, $f'_{(4,1)}(1,1) = 4f'_x(1,1) + f'_y(1,1)$.
- Replace in the previous expression the values of the derivatives and conclude that we obtain a false proposition.
- Conclude that the function is not differentiable at $(1,1)$.
- The statement is FALSE.

d.

- State that if g is C^∞ then, g, g' e g'' are differentiable and consequently continuous in $\mathbb{R}$.
- Conclude that $g'(a) = g'(b) = 0$.
- Check that the conditions to aplly Rolle's Theorem to g and g' at $[a, b]$ are met.
- Use Rolle's Theorem Corollary to conclude that between two consecutive zeros of the derivative, g', there is at most one zero of the function, g.

2.

a.

- Obtain $-7 < u_n < 9$.
- Determine w_n and v_n such that $w_n \leq \frac{u_n}{n^5+1} \leq v_n$, for example $w_n = -\frac{7}{n^5+1}$ and $v_n = \frac{9}{n^5+1}$.

- Obtain $\lim w_n$ and $\lim v_n$, and conclude they are both zero.
- Conclude using the Squeeze Theorem that $lim \frac{u_n}{n^5+1} = 0$.

b.

- Obtain $S_1 = w_1 = \frac{1}{4}$.
- Obtain $S_2 = w_1 + w_2 = \frac{1}{4} + \left(\frac{3}{4}\right)^3$.
- Conclude that $S_1 + S_2 = \frac{1}{2} + \left(\frac{3}{4}\right)^3 > \frac{1}{2}$.

c.

- Obtain $\lim w_n = \left(1 + \frac{-1/2}{n}\right)^n = \frac{1}{\sqrt{e}}$.
- State that, given that $\lim w_n \neq 0$, the series is divergent.

d.

- State that set S has only isolated points, given that its elements' coordinates are given by terms of sequences.
- Obtain $int(S) = \emptyset$.
- Obtain $lim \sqrt[n]{w_n} = 1$.
- Obtain $lim(\ln w_n) = -\frac{1}{2}$.
- Conclude that $S' = \left\{\left(lim \sqrt[n]{w_n}, lim(\ln w_n)\right)\right\} = \left\{\left(1, -\frac{1}{2}\right)\right\}$
- State that S is compact if and only if closed and bounded.
- State that S is not closed because $int(S) \cup \sigma(S) \neq S$ since $\sigma(S) = S \cup S'$.
- Conclude that S is not compact.

e.

- Explain that any values of a, b, c, and d such that, $S \subseteq T$, make $S \cup T$ path connected, thus connected.

3.

a.

- Compute $f_x'(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0^3}{h^2+0^2} - 0}{h} \lim_{h \to 0} \frac{0}{h} = 0$
- Compute $f_y'(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{0^2+h^2} - 0}{h} \lim_{h \to 0} \frac{h}{h} = 1$

b.

- State that f is differentiable in a neighbourhood of $(0{,}0)$ if we can write it as $f(x,y) = f(0{,}0) + f_x'(0{,}0)\,x + f_y'(0{,}0)\,y + R(x,y) = 0 + 0 + y + R(x,y)$, such that
$$\lim_{(x,y)\to(0,0)} \frac{R(x,y)}{\|(x,y)\|} = 0$$
- Obtain $\lim\limits_{(x,y)\to(0,0)} \frac{R(x,y)}{\|(x,y)\|} = \lim\limits_{(x,y)\to(0,0)} \frac{f(x,y)}{\|(x,y)\|} = \lim\limits_{(x,y)\to(0,0)} \frac{\frac{y^3}{x^2+y^2}-y}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \frac{-x^2y}{(x^2+y^2)^{\frac{3}{2}}}$
- Realize that, for example, $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{-x^2y}{(x^2+y^2)^{\frac{3}{2}}} = \lim\limits_{x\to 0} \frac{-x^3}{\left(2x^2\right)^{\frac{3}{2}}} = \lim\limits_{x\to 0} \frac{-x^3}{\left(2x^2\right)^{\frac{3}{2}}} = \lim\limits_{x\to 0} \frac{-x^3}{2\sqrt{2}|x|^3} = \mp\frac{\sqrt{2}}{4} \neq 0$
- Conclude that f is not differentiable at $(0{,}0)$.

c.

- Use the chain rule: $g_x'(x,y) = f_a'(a,b) \times a_x'(x,y) + f_b'(a,b) \times b_x'(x,y) =$
$$= f_a'(x-y, x^2-1) \times 1 + f_b'(x-y, x^2-1) \times 2x$$
- Substitute at the point: $g_x'(1{,}1) = f_a'(0{,}0) \times 1 + f_b'(0{,}0) \times 2 = 0 \times 1 + 1 \times 2 = 2.$

d.

- Use the chain rule: $[g_x'(x,y)]_y' = [f_a'(x-y, x^2-1) + f_b'(x-y, x^2-1) \times 2x]_y' =$
$$= f_{aa}''(x-y, x^2-1) \times a_y'(x,y) + f_{ba}''(x-y, x^2-1) \times a_y'(x,y) \times 2x$$
- Substitute at the point:
$g_{xy}''(1{,}1) = f_{aa}''(0{,}0) \times (-1) + f_{ba}''(0{,}0) \times (-1) \times 2 = 1 \times (-1) + 1 \times (-2) = -4.$

4.

a.

- Looking at the degrees of the functions in the numerator and denominator, 4 and 1 respectively, we suspect that the limit is 0. Alternatively, the iterated and directional limits should be used to believe that the limit can be 0
- Write the limit definition of a function applied to g at $(x,y) = (-2{,}0)$ or, alternatively, at $(a,b) = (0{,}0)$ if you have used a change in the variable ($a = x + 2$ e $b = y$)
- Find a relationship between ε e δ that prove the definition of limit, for example $\varepsilon = \sqrt[3]{\frac{\delta}{4}}$ (or smaller if positive)

b.

- Looking at the first branch of f we can say that all points not belonging to $[-3,-1] \times [-1{,}0]$ will have an image (the problematic point would be (-2,0) but it is excluded from this branch.

- Looking at the second branch of f we can say that all points belonging to $[-3,-1]\times[-1,0]$ will have an image (the problematic point would be (0,0) but it is excluded from this branch.
- So, the domain of f is $\mathbb{R}^2$
- Compute $\frac{(2x+4)^2y^2}{\sqrt{(x+2)^2+y^2}}=0$ and solutions that do not belong to set $[-3,-1]\times[-1,0]$
- Compute $\frac{(2x+4)^2y^2}{\sqrt{(x+2)^2+y^2}}=0$ and choose the solutions that do not belong to set $[-3,-1]\times[-1,0]$
- Compute $\ln\left(\frac{x^2+y^2}{4}\right)=0$ and choose the solutions that belong to set $[-3,-1]\times[-1,0]$
- Geometrically represent the null level curve of f, the set of all points that resulted from the two previous steps

c.

- Explain why f is continuous at $(a,0)$ com $a<-3 \vee a>-1$ (it results from operations between continuous functions in the second branch of the function, more specifically, the composition between a logarithmic function and a rational function)
- Study the continuity of f at points $(a,0)$ with $-3\le a\le -1$ computing $f(a,0)$ and the limit of f at these points:

$$f(a,0)=\ g(a,0)=\ln\left(\frac{a^2}{4}\right)$$

$$\lim_{(x,y)\to(a,0^-)} f(x,y)=\lim_{(x,y)\to(a,0^-)}\ln\left(\frac{x^2+y^2}{4}\right)=\ln\left(\frac{a^2}{4}\right)$$

$$\lim_{(x,y)\to(a,0^+)} f(x,y)=\lim_{(x,y)\to(a,0^+)}\frac{(2x+4)^2y^2}{\sqrt{(x+2)^2+y^2}}=??$$

If $a\neq -2$:

$$\lim_{(x,y)\to(a,0^+)} f(x,y)=\lim_{(x,y)\to(a,0^+)}\frac{(2x+4)^2y^2}{\sqrt{(x+2)^2+y^2}}=\frac{0}{\sqrt{(a+2)^2}}=0$$

If $a=-2$:

$$\lim_{(x,y)\to(a,0^+)} f(x,y)=\lim_{(x,y)\to(a,0^+)}\frac{(2x+4)^2y^2}{\sqrt{(x+2)^2+y^2}}=\frac{0}{0}$$

Using the answer to a) we conclude that this limit is 0. So, f is continuous at $(a,0)$, with $-3\le a\le -1$ only if $a=-2$

Conclude that f is continuous at $(a,0)$ with $a<-3\vee a>-1\vee a=-2$

d.

- Write the general equation of the tangent plane to f at $(-2,0,f(-2,0))$
- Compute the partial derivative of f with respect to x at $(-2,0)$ using the derivation rules. The result

is -1

- Compute the partial derivative of f with respect to y at $(-2,0)$. We can use the derivation rules if $h < 0$ and we must use the definition if $h > 0$. Both are 0
- Find the equation of the tangent plane.