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Resit Exam

- **Date:** May 27, 2021
- **Duration:** 2 hours and 30 minutes
- **Instructions: 1:** This exam has **four questions**. **2:** Write your number and **absolutely nothing else** in this test, and **hand it in at the end. 3:** Answer the exam on the answer book, using the **front** and **back** of each sheet, **stating** which question you are answering, **never** answering **more than one question in the same sheet,** and **not unstapling** any sheet. **4:** If you want to use any sheet of the answer book as space for **drafts,** state it **on the space for the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7: No** individual questions about the exam will be answered. **8:** Break a leg (not literally)!

Number:

1. **(6 points)** Consider the following sets:

$$S = \left\{x \in \mathbb{R} : x = u_n = 3 - \frac{2}{n}(-1)^n, n \in \mathbb{N}\right\}$$
$$T =]3, +\infty[\cap \mathbb{Q}$$

a. **(1 pt)** Let $k = \lim u_n$. Find k, and show that the value you found is indeed the limit, using the definition of limit of a convergent sequence.

b. **(0.75 pts)** State the index p of the first term of the sequence such that $u_p \in B_{0.01}(k)$. Justify.

c. **(1 pt)** Study the series $\sum_{n=1}^{+\infty} \sqrt[n]{u_n}$ regarding convergence.

d. **(1.5 pts)** Let $D = S \cup T$. Find $int(D)$, D' and $\overline{D}$.

e. **(0.75 pts)** State and justify the truth value of the following proposition:

$$\exists m \in (S \cap]3,5[): m \notin T$$

f. **(1 pt)** Let $P = D \cup \{w\} \cup]a, b[$, where $w, a, b \in \mathbb{R}$, $b > a$, $|b - a| = 1$.

It is known that $S' \subset P$, and that P contains no isolated points. Find, and justify, the values of w, a and b.

2. **(4.5 points)** Let $f: \mathbb{R}^2 \to \mathrm{R}_f$ and $g: \mathrm{D_g} \subset \mathbb{R}^2 \to \mathbb{R}^2$ be the functions with expressions:

$$f(x,y) = \begin{cases} \left(e^{x-y}, \sqrt[3]{x+y}\right) & ,(x,y) \neq (0,0) \\ (1,1) & ,(x,y) = (0,0) \end{cases} \qquad g(x,y) = (y^3, 1 + \ln x)$$

a. **(1 pt)** Define de continuity domain of f, and compute $\lim_{(x,y)\to(0,0)} (g \circ f)(x,y)$.

b. **(1.5 pts)** Let $\tilde{f}: \mathbb{R}^2 \to \mathrm{R}_{\tilde{f}}\,, \tilde{f}(x,y) = \left(e^{x-y}, \sqrt[3]{x+y}\right)$. Is the function $\left(g \circ \tilde{f}\right)$ invertible? If so, characterize $\left(g \circ \tilde{f}\right)^{-1}$.

c. **(2 pts)** Consider the function $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \tilde{f}_1(3x^2, x^2) - \left[\tilde{f}_2(4x^3, 4x^3)\right]$. Find $h'(x)$. Then, show that there exists $c \in \left]0, \frac{1}{2}\right[: h'(c) = 0$.

3. **(5 points)** Let $f: D_f \subset \mathbb{R}^2 \to \mathbb{R}$ be the function with expression:

$$f(x,y) = \begin{cases} x - y & , \quad |x| = 2 \\ \dfrac{(x-2)^2}{\sqrt{(x-2)^2 + (y-2)^2}} & , \quad |x| \neq 2 \end{cases}$$

a. **(1 pt)** Find D_f. Define and geometrically represent the level curve of f with level 0.

b. **(1.5 pts)** Study f regarding continuity at points of the form $(2, a), a \in \mathbb{R}$.

c. **(1.5 pts)** Study f regarding differentiability at points of the form $(2, a), a \in \mathbb{R}$.

d. **(1 val)** Consider the sets:

$A = \left\{(x,y) \in D_f : |x| = 2 \wedge y \geq -1\right\}$ $\qquad B = \{(x,y) \in \mathbb{R}^2 : y = kx\,, k \in \mathbb{R}\}$

$C = \bar{B}_z(2,-1), z \in \mathbb{R}^+$

State, and justify, the set of values of k that ensure that $A \cup B$ is connected, and the set of values of z that ensure that $A \cup C$ is path-connected.

4. **(4.5 points)** Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a class C^2 function. It is known that $\nabla_f(-1,1) = (2,-3)$, $\nabla_f(1,-1) = (4,5)$ and $H_f(-1,1) = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. Let $g: \mathbb{R}^2 \to \mathbb{R}$ be the function with expression $g(x,y) = 2f(xy, x) + x^2y$.

a. **(1 pt)** Show that on $[1,2] \times [-4,4]$, g has a global minimum.

b. **(1 pt)** Compute $g'_x(1,-1)$.

c. **(1.5 pts)** Compute $g''_{xy}(1,-1)$.

d. **(1 pt)** Suppose now that f is defined by $f(a,b) = ab$. Compute:

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{\ln(x^2y+1)}$$

Solution Topics

1.

a.

- Conclude that $k = 3$.
- Write the definition of limit of a convergent sequence applied to the context, and find a valid relationship between n and δ, for example, $n > \frac{2}{\delta}$.
- State the index p of the first term of the sequence such that $|u_n - 3| < \delta$, given by $\left\lfloor \frac{2}{\delta} + 1 \right\rfloor$.

b.

- State $\delta = 0.01$.
- Substitute $\delta = 0.01$ in $\left\lfloor \frac{\delta}{2} + 1 \right\rfloor$.
- Conclude that the order is 201.

c.

- Find $\lim \sqrt[n]{u_n}$, computing $\lim \left[(u_n)^{\frac{1}{n}} \right] = \left[\lim u_n \right]^{\lim \frac{1}{n}}$ or, alternatively, $\lim \frac{u_{n+1}}{u_n}$ $(u_n \geq 0)$.
- Conclude that $\lim \sqrt[n]{u_n} = 1 \neq 0$.
- Conclude, by the convergence criterium, that $\sum_{n=1}^{+\infty} \sqrt[n]{u_n}$ is divergent because $\lim \sqrt[n]{u_n} \neq 0$.

d.

- State that $int(D) = \emptyset$.
- State that $D' = [3, +\infty[$.
- State that $\overline{D} = S \cup [3, +\infty[$.

e.

- Conclude that the proposition is false. There are no elements of (u_n) in $]3,5[$ that are irrational numbers. Any term of the given sequence can be written as a division of integers (rational number).

f.

- State that $a = 2$ and $b = 3$ so that P does not have isolated points
- State that $S' = \{3\}$
- State that $k = 3$ to force $S' \subset P$

2.

a.

- Compute $\lim_{(x,y)\to(0,0)} f(x,y) = (1,0)$.
- Conclude that $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$, f is not continuous at $(0,0)$.
- $D_f^{cont} = \mathbb{R}^2 \backslash \{(0,0)\}$.
- Justify the continuity of f with the continuity of each component function
- Compute $\lim_{(x,y)\to(0,0)} (g \circ f)(x,y) = \lim_{(a,b)\to(1,0)} g(a,b) = (0,1)$.

b.

- Show that the function $\left(g \circ \tilde{f}\right)$ is one to one (injective)
- Show that the function $\left(g \circ \tilde{f}\right)$ is surjective
- Find the general expression of the inverse function $\left(g \circ \tilde{f}\right)^{-1}$.
- Characterize the function $\left(g \circ \tilde{f}\right)^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$, $(u,v) = \left(\frac{u+v-1}{2}, \frac{u-v+1}{2}\right)$.

c.

- Show that $h(x) = \tilde{f}_1(3x^2, x^2) - \tilde{f}_2(4x^3, 4x^3) = e^{2x^2} - 2x$.
- Find $h'(x) = 4xe^{2x^2} - 2$.
- Show that h is continuous in $\left[0, \frac{1}{2}\right]$, and that $h'(0) \times h'\left(\frac{1}{2}\right) < 0$.
- So, using the Corollary of the Bolzano Theorem, $\exists c \in \left]0, \frac{1}{2}\right[: h'(c) = 0$.

3.

a.

- Conclude that $D_f = \mathbb{R}^2$.
- Determine $L_f^0 = \{(a,a) \in \mathbb{R}^2 : \ a = 2 \vee a = -2\} = \{(2,2), (-2,-2)\}$.
- Graphically represent the two points that make up the level curve.

b.

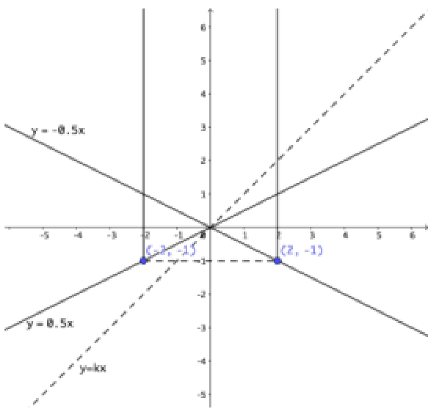
- Compute $f(2,a) = 2 - a$
- Obtain $\lim_{\substack{(x,y)\to(2,a) \\ |x|=2}} f(x,y) = 2 - a$
- Obtain $\lim_{\substack{(x,y)\to(2,a) \\ |x|\neq 2}} f(x,y) = 0 \ \ if \ \ a \neq 2$
- Conclude that the function can only be continuous for $a = 2$.
- Show, using the definition, that f is continuous at $(2,2)$.

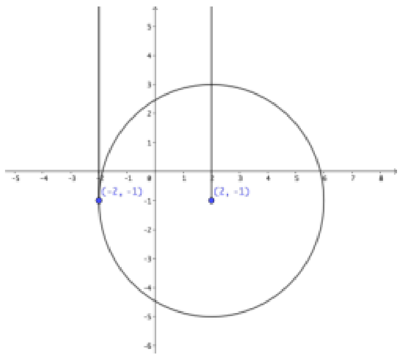
c.

- Justify that if f is not continuous at points $(2, a), a \neq 2$, it cannot be differentiable at those points.
- Study the differentiability of the function at $(2,2)$.
- State the conditions under which f is differentiable at $(2,2)$.
- Compute $\frac{\partial f}{\partial x}(2,2)$ using the definition of partial derivative and conclude that it does not exist.
- State that f is not differentiable at $(2,2)$.

d.

- Graphically represent $A \cup B$ and conclude that this set will be connected for $k \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.
- Graphically represent $A \cup C$ and conclude that this set will be path connected whenever the radius of the closed ball is at least 4.





4.

a.

- State that the set $[1,2] \times [-4,4]$ is compact (bounded and closed)
- Justify why the set is closed and bounded
- State that f is of class C^2 in $\mathbb{R}^2$, so it is continuous in $\mathbb{R}^2$. State that g is continuous in $\mathbb{R}^2$ because it results from operations between continuous functions in $\mathbb{R}^2$.
- State that g is continuous in $[1,2] \times [-4,4]$. Refer the Extreme Value Theorem (Weierstrass) that forces the existence of a maximum and a minimum of g in $[1,2] \times [-4,4]$.

b.

- Use the Chain Rule to compute $g'_x(1,-1)$:

$$g'_x(1,-1) = \left[2\left[\frac{\partial f}{\partial a}(xy,x)\frac{\partial a}{\partial x}(x,y) + \frac{\partial f}{\partial b}(xy,x)\frac{\partial \mathrm{b}}{\partial x}(x,y)\right] + 2xy\right]_{(1,-1)} =$$
$$= (2[f'_a(xy,x)y + f'_b(xy,x) * 1] - 2)_{(1,-1)} = 2[2(-1) + (-3)] - 2 = -12$$

c.

- Use the Chain Rule to compute $g''_{xy}(1,-1)$:

$$g''_{xy}(1,-1) = \left(\left[2\left[\frac{\partial f}{\partial a}(xy,x) * y + \frac{\partial f}{\partial b}(xy,x)\right] + 2xy\right]'_y\right)_{(1,-1)} = 2\left[\left[\frac{\partial^2 f}{\partial a^2}(xy,x)\frac{\partial a}{\partial y}(x,y) * y + + \frac{\partial f}{\partial a}(xy,x) * 1 + \frac{\partial^2 f}{\partial b \partial a}(xy,x)\frac{\partial a}{\partial y}(x,y)\right] + 2x\right]_{(1,-1)} = 2[f''_{aa}(-1,1) * 1 * (-1) + f'_a(-1,1) + f''_{ba}(-1,1) * 1] \quad + 2 = 2[-1+2+2] \quad + 2 = 8$$

d.

- Write $\lim\limits_{(x,y)\to(0,0)} \frac{g(x,y)}{\ln(x^2y+1)} = \lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{\ln(x^2y+1)}$.
- Change of variable $z = x^2y$.
- $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{\ln(x^2y+1)} = 3\lim\limits_{z\to 0} \frac{z}{\ln(z+1)} = 3\left(\lim\limits_{z\to 0} \frac{\ln(z+1)}{z}\right)^{-1} = 3 * 1 = 3.$

or

- Use the Cauchy Rule to find $3\lim\limits_{z\to 0} \frac{z}{\ln(z+1)}$
- Write $3\lim\limits_{z\to 0} \frac{z}{\ln(z+1)} = 3\lim\limits_{z\to 0} \frac{1}{\frac{1}{z+1}} = 3.$