Nova School of Business and Economics
2020 – 2021 S1
Calculus I
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Midterm

- **Date:** November 28, 2020
- **Duration:** 2 hours
- **Instructions: 1:** The test has **four questions. 2:** Write your number and **absolutely nothing else** in this test, and **hand it in at the end. 3:** Answer the test on the answer book, using the **front** and **back** of each sheet, **stating** which question you are answering to, **never** answering **more than one question in the same sheet**, and **not unstapling** any sheet. **4:** If you want to use any sheet of the answer book as space for **drafts,** state it **on the space for the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7:** If, on question 4, you answer to all parts, **only the first five** will be graded. **8: No** individual questions about the exam will be answered. **9:** Break a leg (not literally)!

Nº:

1. **(5 pts)** Consider $k, r \in \mathbb{R}^+$, and the following sets:

$$A = [0,k]^3 \qquad B = B_r(0, 2k, 0) \qquad C = \left\{(x,y,z) \in \mathbb{R}^3 : \frac{x}{y} \notin \mathbb{R}\right\}$$

$$D = C \cup (A \setminus]0,k[^3) \qquad E = (A^c \cap B^c)^c \cup \big(C \backslash (C \cap C^c)\big)$$

a. **(1 pt)** Geometrically represent A, B and C in a single cartesian plane, for $r = \frac{k}{2}$.

b. **(1 pt)** State and justify the set of values of r, depending on k, such that $A \subset B$ and, if it exists, the minimum of this set.

c. **(1.5 pts)** State and justify the interior of D, whether D is bounded, and whether D is convex.

d. **(1.5 pts)** State and justify the set of values of r, depending on k, such that E is connected and $A \setminus B = A$.

2. **(5 pts)** Consider $\delta, \gamma \in \mathbb{R}^+$, and the sequence (u_n), defined by:

$$u_n = \begin{cases} 4 + \dfrac{1}{n} & \text{if} \quad n \text{ odd} \\ \dfrac{9}{n^2 - 1} & \text{if} \quad n \text{ even} \end{cases}$$

a. **(1 pt)** State and justify the limit of the sequence of partial sums of (u_n).

b. **(0.5 pts)** Based on your answer to **a.**, classify $\sum_{n=1}^{+\infty} u_n$ regarding convergence.

c. **(2 pts)** State and justify one order from which, simultaneously, all odd order terms of (u_n) are in $B_\delta(4)$ and all even order terms of (u_n) are in $B_\gamma(0)$.

d. **(1.5 pts)** Find the infimum of $A = \{k \in \mathbb{R}: |1 - 2u_n| \leq k, \forall n \in \mathbb{N}\}$.

3. **(5 pts)** Consider the function $f: D \subset \mathbb{R}^2 \to \mathbb{R}^2$, defined by:

$$f(x,y) = \left(\ln(4 - x^2 - y^2), \frac{1}{x^2 + y^2}\right)$$

a. **(1 pt)** Define and geometrically represent D.

b. **(0.5 pts)** Define the boundary and the derived set of D.

c. **(1 pt)** Define the range of f_1 and the range of f.

d. **(1.5 pts)** Define and geometrically describe the general level curve of f_1, and geometrically represent, if possible, the ones of level 0 and 2, and the one which contains $\left(0, \sqrt{4 - e}\right)$.

e. **(1 pts)** Consider the sequence (u_n), defined by $u_n = f_2\left(\frac{1}{n}, \frac{1}{n}\right)$. Compute $\lim (u_n)^{\frac{1}{n}}$.

4. **(5 pts)** State and justify the truth value of **five** of the following **six** propositions:

a. **(1 pt)** If A is a nonempty set, then $\forall x \in A, \exists B \subset A: \left(\{\{x\}, B\} \text{ is a partition of } A\right)$.

b. **(1 pt)** There is no subset of $\mathbb{R} \setminus \mathbb{Q}$ with an infinite cardinal number and empty derived set.

c. **(1 pt)** If (a_n) and (b_n) are sequences such that $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ are divergent, and (u_n) is defined by $u_n = \begin{cases} a_n & \text{if} \quad n \text{ odd} \\ b_n & \text{if} \quad n \text{ even} \end{cases}$, then $\sum_{n=1}^{+\infty} u_n$ is divergent.

d. **(1 pt)** The application which turns each strictly increasing and bounded sequence into the accumulation points of the set of its terms is a function.

e. **(1 pt)** If (u_n) is a strictly decreasing sequence and $f: \mathbb{N} \to \mathbb{R}$ is the function defined by $f(n) = u_n$, then the level sets of f are compact.

f. **(1 pt)** The graph of a function $f: \mathbb{R}^2 \to \mathbb{R}$ may be symmetric relatively to any of the following planes: XOY, XOZ and YOZ.

Solution Topics

1.

a.

- Geometrically represent A, the cube defined by the vertices $(0,0,0)$, $(k,0,0)$, $(0,k,0)$ and $(0,0,k)$ and the region it bounds
- Geometrically represent B, the interior of the ball centered at $(0,2k,0)$ and with radius $\frac{k}{2}$
- Geometrically represent C, the XOZ plane

b.

- Explain that $A \subset B$ if and only if the radius of B is greater than the distance between the center of B, $(0,2k,0)$, and the point in A which is furthest away from it, $(k,0,k)$
- Show that the set of values of r is $R = \left]\sqrt{6}k, +\infty\right[$
- State that the infimum of R is $\sqrt{6}k$
- State that $\sqrt{6}k \notin R$, therefore R has no minimum

c.

- Explain that D is the union of the XOZ plane with the cube defined by the vertices $(0,0,0)$, $(k,0,0)$, $(0,k,0)$ and $(0,0,k)$
- Explain that all open balls centered at points in D are regions bounded by spheres with a positive radius and, so, always contain points in D^c, which means that $\operatorname{int}(D) = \emptyset$
- Explain that D contains the XOZ plane, which is unbounded, which means that D is also unbounded
- Show that, for example, $(0,0,0), (0,k,k) \in D$, but $\frac{1}{2}(0,0,0) + \frac{1}{2}(0,k,k) \notin D$, which means that D is not convex

d.

- Show that $E = A \cup B \cup C$
- Explain that E is connected if and only if $r \geq k$
- Explain that $A \setminus B = A$ if and only if $r \leq k$
- Conclude that the set of values of k is $\{k\}$

2.

a.

- Explain that the sequence of partial sums of (u_n) is the sum of the sequences of partial sums of the subsequences of (u_n) with its odd and even order terms, (v_n) and (w_n), respectively
- Show that $\lim(v_n) = 4$, which means that the sequence of its partial sums approaches $+\infty$
- Show that of $\lim(w_n) = 0$, which means that, as (w_n) is positive, the sequence of its partial sums approaches $b \in \mathbb{R}^+$ or $+\infty$
- Conclude that the limit of the sequence of partial sums of (u_n) is $+\infty$

b. Explain that, as the limit of the sequence of partial sums of (u_n) is $+\infty$, $\sum_{n=1}^{+\infty} u_n$ is divergent

c.

- Show that $|v_n - 4| < \delta \Leftrightarrow n > \frac{1}{\delta}$
- Show that $|w_n - 0| < \gamma \Leftrightarrow n > \sqrt{\frac{\gamma}{9} + 1}$
- Explain that $\begin{cases} |v_n - 4| & < \delta \\ |w_n - 0| & < \gamma \end{cases} \Leftrightarrow n > \max\left\{\frac{1}{\delta}, \sqrt{\frac{\gamma}{9} + 1}\right\}$
- Conclude that a possible first order is $\left\lfloor \max\left\{\frac{1}{\delta}, \sqrt{\frac{\gamma}{9} + 1}\right\} + 1 \right\rfloor$

d.

- Show that, if V is the set of terms of (v_n), V has no minimum, and the infimum and maximum of V are, respectively, 4 and 5
- Show that, if W is the set of terms of (w_n), W has no minimum, and the infimum and maximum of W are, respectively, 0 and 3
- Conclude that, if U is the set of terms of (u_n), U has no minimum, and the infimum and maximum of U are, respectively, 0 and 5
- Show that, if T is the set of terms of $(|1 - 2u_n|)$, the maximum of T is 9
- Explain why $A = [9, +\infty[$
- State that the infimum of A is 9

3.

a.

- Show that $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\} \setminus \{(0,0)\}$
- Geometrically represent D, the interior of the disk centered at $(0,0)$ and with radius 2, subtracted of $(0,0)$

b.

- State that $\mathrm{fr}(D) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \in \{0,4\}\}$
- State that $D' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$

c.

- Show that $R_{f_1} =]-\infty, \ln 4[$
- Show that $\forall (a,b) \in R_f, b = \frac{1}{4-e^a}$
- Conclude that $R_f = \left\{\left(a, \frac{1}{4-e^a}\right) \in \mathbb{R}^2 : a < \ln 4\right\}$

d.

- Show that $\forall k < \ln 4, C_{f_1}^k = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 4 - e^k\}$, the circle centered at $(0,0)$ and with radius $\sqrt{4-e^k}$
- Geometrically represent $C_{f_1}^0$, the circle centered at $(0,0)$ and with radius $\sqrt{3}$
- Explain why $2 \notin CD_{f_1}$, which implies that $\nexists C_{f_1}^2$
- Show that $f_1\left(0, \sqrt{4-e}\right) = 1$
- Geometrically represent $C_{f_1}^1$, the circle centered at $(0,0)$ and with radius $\sqrt{4-e}$

e.

- Show that $u_n = \frac{n^2}{2}$
- Show that $\lim \frac{u_{n+1}}{u_n} = 1$
- Conclude that $\lim (u_n)^{\frac{1}{n}} = 1$

4.

a. State that the proposition is true because $\{x, A \setminus \{x\}\}$ is a partition of A

b. State that the proposition is false because, for example, $A = \{\pi n \in \mathbb{R} : n \in \mathbb{N}\} \subset (\mathbb{R} \setminus \mathbb{Q})$ has an infinite cardinal number and empty derived set

c. State that the proposition is false because, for example, if $a_n = \begin{cases} 0 & \text{if} \quad n \text{ odd} \\ n & \text{if} \quad n \text{ even} \end{cases}$ and $b_n = \begin{cases} n & \text{if} \quad n \text{ odd} \\ 0 & \text{if} \quad n \text{ even} \end{cases}$, then $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ are divergent, but $\sum_{n=1}^{+\infty} u_n$ is convergent

d. State that the proposition is true because the sets of terms of each strictly increasing and bounded sequence has a single accumulation point, which means that the application turns each argument into a single image

e. State that the proposition is true because each level set of f has a single point, which means it is closed and bounded

f. State that the proposition is true because, for example, the graph of $f: \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = 0$, is symmetric relatively to the XOY, XOZ and YOZ planes