Nova School of Business and Economics
2020 – 2021 S1
Calculus I
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Resit Exam

- **Date:** January 25, 2021
- **Duration:** 2 hours and 30 minutes
- **Instructions: 1:** The exam has **four questions. 2:** Write your number and **absolutely nothing else** in this exam, and **hand it in at the end. 3:** Answer the exam on the answer book, using the **front** and **back** of each sheet, **stating** which question you are answering to, **never** answering **more than one question in the same sheet**, and **not unstapling** any sheet. **4:** If you want to use any sheet of the answer book as space is **drafts,** state it **on the space is the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7:** If, in question 4, you answer to all parts, only the **first five** will be graded. **8: No** individual questions about the exam will be answered. **9:** Break a leg (not literally)!

Nº:

1. **(5 pts)** Consider the sequence (u_n), defined by:

$$u_n = \frac{2n-5}{n}$$

a. **(1 pt)** Compute $\lim u_n$ and confirm your answer using the definition of limit of a sequence.

b. **(1 pt)** Classify $\sum_{n=1}^{+\infty}(u_n - 1)^{2n}$ regarding convergence.

c. **(1 pt)** Consider a sequence (v_n), such that $|v_n| \leq |u_n|, \forall n \in \mathbb{N}$. State and justify the truth value of the following propositions:

(i) **(0.5 pts)** (v_n) is necessarily bounded.

(ii) **(0.5 pts)** (v_n) is necessarily convergent.

d. **(2 pts)** Consider the sequence (w_n), defined by $w_n = u_{n+1} - u_n$.

(i) **(0.5 pts)** Show that (w_n) is monotone.

(ii) **(1.5 pts)** Consider the set $A = \{x \in \mathbb{R}: x = u_n \wedge n \in \mathbb{N}\}$, and the sequence (a_n) which gives, for every $n \in \mathbb{N}$, the highest value of $r \in \mathbb{R}^+$ such that $B_r(u_n) \cap A = \{u_n\}$. Find, if they exist, the *infimum*, *supremum*, *minimum* and *maximum* of the set of terms of (a_n).

2. **(5.5 pts)** Consider the functions $f: D_f \subset \mathbb{R}^2 \to R_f \subset \mathbb{R}^3$ and $g: \mathbb{R}^3 \to \mathbb{R}^2$, defined, respectively, by:

$$f(x,y) = (f_1(x,y), f_2(x,y), f_3(x,y)) = \left(ln(y - x^2), \frac{1}{x^2 - y + 1}, \sqrt[3]{2y - 2x^2}\right) \qquad g(u,v,w) = (u,v)$$

a. **(0.5 pts)** Define and geometrically represent D_f.

b. **(1.5 pts)** State and justify whether D_f is open, whether it is bounded, whether it is connected and whether it is path – connected.

c. **(0.75 pts)** Show that $\sum_{i=1}^{3} f_i$ has a global maximum in $[-a, a] \times [a^2 + 1 + \varepsilon, a^2 + 1 + 2\varepsilon]$, $\forall a \in \mathbb{R}$ and $\varepsilon > 0$.

d. **(0.75 pts)** Define the general level curve of f_3 and geometrically represent, if possible, the ones of levels 0, $\sqrt[3]{2}$ and $\sqrt[3]{4}$.

e. **(1.5 pts)** Find, if it exists, the general expression of $f \circ g$, and define its domain ~~and range~~.

f. **(0.5 pts)** State and justify whether $f \circ g$ is invertible.

3. **(5 pts)** Consider the function $f: D_f \subset \mathbb{R}^2 \to \mathbb{R}$, defined by:

$$f(x,y) = \frac{x^2 y \sqrt{1 - x^2 - y^2}}{x^2 + y^2}$$

a. **(0.5 pts)** Define and geometrically represent D_f.

b. **(1.5 pts)** State and justify whether f is continuously extendable to $(0,0)$, and, if so, characterize F, the function which results from this extension, stating its domain, codomain, and general expression.

c. **(1 pt)** Compute, if it exists, $\nabla_F(0,0)$.

d. **(0.75 pts)** Compute, if it exists, $F'_{(1,1)}(0,0)$.

e. **(1.25 pts)** State and justify whether F is differentiable at $(0,0)$.

4. **(4.5 pts)** State and justify the truth value of **five** of the following **six** propositions:

a. **(0.9 pts)** Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a function such that $\lim_{x \to a} f(x) \neq b$, and A be the set of paths T such that $\lim_{\substack{x \to a \\ x \in T}} f(x) = b$. Then, the cardinal number of A is 0.

b. **(0.9 pts)** Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}$. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a continuous function at a such that $f(a) = b$, and $g: \mathbb{R}^n \to \mathbb{R}$ be the function whose graph results from a vertical translation of the graph of f, b units down. Then, $\lim_{x \to a} g(x) = 0$.

c. **(0.9 pts)** Let $f: A \subset \mathbb{R}^2 \to \mathbb{R}$ be a function with range B. Then, the set of points of the graph of f is $A \times B$.

d. **(0.9 pts)** Let (u_n) be a geometric progression with first term u_1 and common ratio r. Let $f: \mathbb{R} \setminus \{0\} \times]-1,1[\to \mathbb{R}$ be the function which turns each (u_1, r) into the sum of $\sum_{n=1}^{+\infty} u_n$. Then, $f''_{rr}\left(1, \frac{1}{2}\right) = 16$.

e. **(0.9 pts)** Let $a \in \mathbb{R}^n$. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a differentiable function at a, and $g: \mathbb{R}^n \to \mathbb{R}$ be the function whose graph is the tangent plane to the graph of f at $\left(a, f(a)\right)$. Then, the zero – level set of $f - g$ is $\{a\}$.

f. **(0.9 pts)** Let $f: \mathbb{R}^n \to \mathbb{R}^n$ and $g: \mathbb{R}^n \to \mathbb{R}^n$ be two invertible functions. Then, $g \circ f$ is invertible.

Solution Topics

1.

a.

- Show, using limit rules, that $\lim u_n = 2$
- Show, using the definition of limit of a convergent sequence, that $\lim u_n = 2$
- Use in the proof, for example, the fact that, for every $\delta > 0$, all terms of (a_n) whose order is greater than or equal to the first natural number greater than $\frac{5}{\delta}$ distance less than δ from 0

b.

- Show that $b_n = (u_n - 1)^{2n} = \left(\left(1 - \frac{5}{n}\right)^n\right)^2$
- Show that $\lim b_n = \frac{1}{e^{10}}$
- Conclude that, as $\lim b_n \neq 0$, $\sum_{n=1}^{+\infty} b_n$ is divergent

c.

(i)

- Show, for example, that $\forall n \in \mathbb{N}, |u_n| \leq 3$, which implies that (u_n) is bounded
- Explain why (v_n) is bounded
- Conclude that the proposition is true

(ii)

- Show that, if (v_n) is defined by, for example, $v_n = \begin{cases} 0 & \text{if} \quad n \leq 5 \\ (-1)^n & \text{if} \quad n > 5 \end{cases}$, then $\forall n \in \mathbb{N}, |v_n| \leq |u_n|$, and (v_n) is divergent
- Conclude that the proposition is false

d.

(i)

- Show, for example, that $\forall n \in \mathbb{N}, w_{n+1} - w_n < 0$
- Conclude that (w_n) is strictly decreasing

(ii)

- Explain why (a_n) is defined by $a_n = |u_{n+1} - u_n| = w_n$, which implies that is strictly decreasing
- Show that $a_1 = \frac{5}{2}$ and that $\lim a_n = 0^+$
- Explain why, if B is the set of terms of (a_n), $\inf(B) = 0$, $\sup(B) = \frac{5}{2}$, $\nexists \min(B)$ and $\max(B) = \frac{5}{2}$

2.

a.

- Show that $D_f = \{(x, y) \in \mathbb{R}^2 : y > x^2 \wedge y \neq x^2 + 1\}$
- Geometrically represent D_f

b.

- State that $\text{int}(D_f) = D_f$
- State that, as $\text{int}(D_f) = D_f$, D_f is open
- State, for example, that $A = \{(x, y) \in \mathbb{R}^2 : x = 0 \wedge y > 1\}$ is a semi – straight line, therefore unbounded, contained in D_f, which implies that D_f is unbounded
- Show that D_f is disconnected, using, for example, the sets $D_f^1 = \{(x, y) \in \mathbb{R}^2 : x^2 < y < x^2 + 1\}$ and $D_f^2 = \{(x, y) \in \mathbb{R}^2 : y > x^2 + 1\}$
- Show that D_f is not path – connected, using the fact that it is disconnected or, for example, the points $\left(0, \frac{1}{2}\right)$ and $(0,2)$

c.

- Explain why $h = f_1 + f_2 + f_3$ is a continuous function
- Show that $B = [-a, a] \times [a^2 + 1 + \varepsilon, a^2 + 1 + 2\varepsilon] \in D_h$
- Show that B is compact
- State that Weierstrass' theorem applies to h in B, which implies that h has a global maximum in B

d.

- Show that $\forall k \in \mathbb{R}^+ \setminus \{\sqrt[3]{2}\}, C_f^k = \left\{\left(x, x^2 + \frac{k^3}{2}\right) \in \mathbb{R}^2 : x \in \mathbb{R}\right\}$
- Explain why $0 \notin CD_{f_3}$, which implies that $\nexists C_{f_3}^0$
- Explain why $\sqrt[3]{2} \notin CD_{f_3}$, which implies that $\nexists C_{f_3}^{\sqrt[3]{2}}$
- Show that $C_{f_3}^{\sqrt[3]{4}} = \{(x, x^2 + 2) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ and geometrically represent it

e.

- Show that $D_{f \circ g} = \{(u, v, w) \in \mathbb{R}^3 : v > u^2 \wedge v \neq u^2 + 1\}$
- Show that $\forall (u, v, w) \in D_{f \circ g}, (f \circ g)(u, v) = \left(\ln(v - u^2), \frac{1}{u^2 - v + 1}, \sqrt[3]{2v - 2u^2}\right)$
- Show that $CD_{(f \circ g)_3} = \mathbb{R}^+ \setminus \{\sqrt[3]{2}\}$
- Show that $\forall (b, c, d) \in CD_{f \circ g}, b = \ln\frac{d^3}{2} \wedge c = \frac{2}{2 - d^3}$
- Conclude that $CD_{f \circ g} = \left\{\left(\ln\frac{d^3}{2}, \frac{2}{2 - d^3}, d\right) \in \mathbb{R}^2 : d \in \mathbb{R}^+ \setminus \{\sqrt[3]{2}\}\right\}$

f.

- Show that, for example, $(f \circ g)(0,2,0) = (f \circ g)(0,2,1)$, which implies that $f \circ g$ is not injective
- Conclude that $f \circ g$ is not invertible

3.

a.

- Show that $D_f = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \wedge (x, y) \neq (0,0)\}$
- Geometrically represent D_f

b.

- Show, using the definition of limit of a function, that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$
- Use in the proof, for example, the fact that, for every $\delta > 0$, all arguments of f which distance less than δ from $(0,0)$ have images which distance less than δ of 0
- Conclude that $\lim_{(x,y)\to(0,0)} f(x,y) \in \mathbb{R}$, which implies that f is continuously extendable to $(0,0)$
- State that the characterization of F is $\begin{array}{l} F: \mathbb{R}^2 \to \mathbb{R}^2 \\ F(x,y) = \begin{cases} f(x,y) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases} \end{array}$

c.

- Show, using the definition of partial derivative, that $F'_x(0,0) = 0$
- Show, using the definition of partial derivative, that $F'_y(0,0) = 0$
- Conclude that $\nabla_F(0,0) = (0,0)$

d.

- Show, using the definition of directional derivative, that $F'_{(1,1)}(0,0) = \frac{1}{2}$

e.

- Show that $F(0,0) + F'_x(0,0)(x-0) + F'_y(0,0)(y-0) = 0$
- Show, for example, that $\lim_{\substack{(x,y)\to(0,0) \\ y=x \\ x>0}} \frac{F(x,y)-0}{\|(x,y)-(0,0)\|} = \frac{\sqrt{2}}{4}$
- Conclude that $\lim_{(x,y)\to(0,0)} \frac{F(x,y)-0}{\|(x,y)-(0,0)\|} \neq 0$, which implies that F is not differentiable at $(0,0)$
- Alternatively, show that $F'_{(1,1)}(0,0) \neq \nabla_F(0,0) \cdot (1,1)$, which implies that F is not differentiable at $(0,0)$

4.

a. State that the proposition is false because, for example, if $a = (0,0)$, $b = \frac{1}{2}$ and $f: \mathbb{R}^2 \to \mathbb{R}$ is the function defined by $f(x,y) = \frac{xy}{x^2+y^2}$, then $\nexists \lim_{(x,y)\to a} f(x,y)$ and $\{(x,x) \in \mathbb{R}^2 : x \neq 0\} \in T$

b. State that the proposition is true because g is defined by $g(x) = f(x) - b$, g is continuous at a and $g(a) = 0$

c. State that the proposition is false because, for example, if $f: \mathbb{R}^2 \to \mathbb{R}$ is the function defined by $f(x,y) = x + y$, then $A = \mathbb{R}^2$, $B = \mathbb{R}$, $A \times B = \mathbb{R}^3$ and $(1,2,4) \notin \text{Graph}_f$

d. State that the proposition is true because, as $r \in \,]-1,1[$, f is defined by $f(u_1, r) = \frac{u}{1-r}$, which implies that $f''_{rr}(u_1, r) = \frac{2u_1}{(1-r)^3}$

e. State that the proposition is false because, for example, if $a = 1$ and $f: \mathbb{R} \to \mathbb{R}$ is the function defined by $f(x) = x$, then f is differentiable at a, $g = f$ and $C^0_{f-g} = \mathbb{R}$

f. State that the proposition is true because different arguments of $g \circ f$ have different images and $CD_{g\circ f} = \mathbb{R}^n$, which implies that $g \circ f$ is bijective