Nova School of Business and Economics
2020 – 2021 S1
Calculus I
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Regular Exam

- **Date:** January 5, 2021
- **Duration:** 2 hours and 30 minutes
- **Instructions: 1:** The exam has **four questions**. **2:** Write your number and **absolutely nothing else** in this exam, and **hand it in at the end**. **3:** Answer the exam on the answer book, using the **front** and **back** of each sheet, **stating** which question you are answering to, **never** answering **more than one question in the same sheet**, and **not unstapling** any sheet. **4:** If you want to use any sheet of the answer book as space for **drafts**, state it **on the space for the question number. 5: Show** all your work. **6: No** written support or calculators are allowed. **7:** If, in question 4, you answer to all parts, only the **first five** will be graded. **8: No** individual questions about the exam will be answered. **9:** Break a leg (not literally)!

Nº:

1. **(5 pts)** Consider $x \in \mathbb{R}$, and the sequences (a_n), (b_n) and (c_n), defined, respectively, by:

$$\cancel{a_n = \frac{1}{\sqrt{2n+3}}} \qquad \cancel{b_n = \frac{1}{\sqrt[n]{2n+3}}} \qquad c_n = \frac{(4x-1)^n}{2^{n+1}}$$

a. **(1 pt)** Compute $\lim a_n$ and confirm your answer using the definition of limit of a sequence.

b. **(1 pt)** Compute $\lim b_n$ and $\lim \frac{b_n}{a_n}$.

c. **(0.75 pts)** Compute $\lim \left(\frac{a_n + b_n}{b_n}\right)^{\frac{b_n}{a_n}}$.

d. **(2.25 pts)** Define the set of values of x such that:

(i) **(1.25 pts)** $\sum_{n\geq 1} c_n$ is convergent.

(ii) **(1 pt)** The limit of the sequence of partial sums of (c_n) is $-\frac{1}{6}$.

2. **(5.5 pts)** Consider the functions $f: D_f \subset \mathbb{R}^2 \to \mathbb{R}^2$ and $g: D_g \subset \mathbb{R}^2 \to \mathbb{R}^2$, defined, respectively, by:

$$f(x,y) = \left(f_1(x,y), f_2(x,y)\right) = \left(ln\left(1+\frac{x}{y}\right), \frac{1}{x-y}\right) \qquad g(u,v) = \left(g_1(u,v), g_2(u,v)\right) = \left(e^u - 1, \frac{1}{v}\right)$$

a. **(1 pts)** Define and geometrically represent D_f.

b. **(0.75 pts)** Define the interior, the boundary, and the derived set of D_f.

c. **(1 pt)** State and justify whether D_f is open, whether it is closed, ~~whether it is connected~~ and whether it is convex.

d. **(1 pt)** Define the general level curve of f_1 and geometrically represent, if possible, the ones of levels 0, 1 and 2.

e. **(1.25 pts)** Characterize $g \circ f$, stating its domain, codomain and general expression.

f. **(0.5 pts)** Show that the range of $g \circ f$ is not $\mathbb{R}^2$.

3. **(5 pts)** Consider the functions $f: D \subset \mathbb{R}^2 \to \mathbb{R}^2$ and $g: D \subset \mathbb{R}^2 \to \mathbb{R}$, defined, respectively, by:

$$f(x,y) = \left(f_1(x,y), f_2(x,y)\right) = \left(\frac{x^2y^2}{x^2+y^2}, \frac{x^2+1}{4}\right) \qquad g(x,y) = \begin{cases} f_1(x,y) & \text{if} \quad y \le x^2 \\ f_2(x,y) & \text{if} \quad y > x^2 \end{cases}$$

a. **(1.5 pts)** State and justify whether f is continuously extendable to $(0,0)$, and, if so, characterize h, the function which results from this extension, stating its domain, codomain, and general expression.

b. **(1.5 pts)** State and justify whether g is differentiable at $(1,1)$.

c. **(1 pt)** Consider $a \in \mathbb{R}$, $b \in \mathbb{R}^+$ and $v \in \mathbb{R} \setminus \{0\}$. With no calculations, state and justify what the value of $g'_{(0,v)}(a, a^2+b)$ is.

d. **(1 pt)** Consider the set $A = \left\{\left(\frac{1}{n}, \frac{2}{n}\right) \in \mathbb{R}^2 : n \in \mathbb{N}\right\} \cup (\mathbb{R}^2 \setminus D)$, and the function $i: \mathbb{R}^2 \to \mathbb{R}$, differentiable at $(0,0)$. Show that i has a global minimum and a global maximum in A.

4. **(4.5 pts)** State and justify the truth value of **five** of the following **six** propositions:

a. **(0.9 pts)** If (u_n) is a sequence of positive terms and $\sum_{n=1}^{+\infty} u_n$ is convergent, then $\forall b \in \mathbb{R}^+, \exists n \in \mathbb{N}: u_n < b$.

b. **(0.9 pts)** If $f: \mathbb{R}^n \to \mathbb{R}^n$ is an invertible function and f^{-1} is the inverse of f, then $f^{-1} \neq f$.

c. **(0.9 pts)** The interior of the set of points of the graph of a function $f: \mathbb{R}^2 \to \mathbb{R}$ is empty.

d. **(0.9 pts)** The continuity domain of $f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$, is $\mathbb{R}$.

e. **(0.9 pts)** If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^p \to \mathbb{R}^q$ are two functions such that $f \circ g$ and $g \circ f$ exist, then the jacobian matrices of $f \circ g$ and $g \circ f$ have the same dimensions.

f. **(0.9 pts)** If $a \in \mathbb{R}^n$, Ω is the set of functions with domain $\mathbb{R}^n$ and codomain $\mathbb{R}$, A is the subset of Ω containing the functions which are differentiable at a, and B is the subset of Ω containing the functions such that at least one partial derivative at a is not a real number, then $\{A, B\}$ is a partition of Ω.

Solution Topics

1.

a.

- Show, using limits rules, that $\lim a_n = 0$
- Show, using the definition of limit of a convergent sequence, that $\lim a_n = 0$
- Use in the proof, for example, the fact that, for every $\delta > 0$, all terms of (a_n) whose order is greater than or equal to the first natural number greater than $\frac{1}{2\delta^2} - \frac{1}{2}$ distance less than δ from 0

b.

- Show that $b_n = \sqrt[n]{d_n}$, with $d_n = \frac{1}{2n+3}$
- Show, using limit rules or the definition of limit of a convergent sequence, that $\lim \frac{d_{n+1}}{d_n} = 1$
- Conclude that $\lim b_n = 1$
- Explain why, as $\lim b_n = 1$ and $\lim a_n = 0^+$, $\lim \frac{b_n}{a_n} = +\infty$

c.

- Show that $\lim \left(\frac{a_n + b_n}{b_n}\right)^{\frac{b_n}{a_n}} = \lim \left(1 + \frac{1}{\frac{b_n}{a_n}}\right)^{\frac{b_n}{a_n}}$
- Explain that, as $\lim \frac{b_n}{a_n} = +\infty$, $\lim \left(1 + \frac{1}{\frac{b_n}{a_n}}\right)^{\frac{b_n}{a_n}} = e$

d.

(i)

- Show that (c_n) is a geometric progression whose first term and common ratio are, respectively, $\frac{4x-1}{4}$ and $\frac{4x-1}{2}$
- State that $\sum_{n\geq 1} c_n$ is convergent if and only if $\left|\frac{4x-1}{2}\right| < 1$
- Conclude that that happens if and only if $x \in \left]-\frac{1}{4}, \frac{3}{4}\right[$

(ii)

- State that, if $\sum_{n\geq 1} c_n$ is convergent, and only in this case, the limit of the sequence of the partial sums of (c_n) is a real number, the sum of $\sum_{n\geq 1} c_n$
- State that, if $\sum_{n\geq 1} c_n$ convergent, and only in this case, the sum of $\sum_{n\geq 1} c_n$ is $\frac{\frac{4x-1}{4}}{1-\frac{4x-1}{2}}$
- Conclude that the sum of $\sum_{n\geq 1} c_n$ is $-\frac{1}{6}$ if and only if $x \in \{0\}$

2.

a.

- Show that $D_f = \{(x,y) \in \mathbb{R}^2 : ((y > 0 \land y > -x) \lor (y < 0 \land y < -x)) \land y \neq x\}$
- Geometrically represent D_f

b.

- State that $\text{int}(D_f) = D_f$
- State that $\text{fr}(D_f) = \{(x,y) \in \mathbb{R}^2 : y \in \{-x, 0, x\}\}$
- State that $D_f{}' = \{(x,y) \in \mathbb{R}^2 : (y \geq 0 \land y \geq -x) \lor (y \leq 0 \land y \leq -x)\}$

c.

- State that, as $\text{int}(D_f) = D_f$, D_f is open
- State that, as $\overline{D_f} = D_f{}' \neq D_f$, D_f is not closed
- Show that D_f is disconnected, using, for example, the sets $D_f^1 = \{(x,y) \in \mathbb{R}^2 : ((y > 0 \land y > -x) \lor (y < 0 \land y < -x)) \land y < x\}$ and $D_f^2 = \{(x,y) \in \mathbb{R}^2 : ((y > 0 \land y > -x) \lor (y < 0 \land y < -x)) \land y > x\}$
- Show that D_f is not convex, using the fact that it is disconnected or, for example, the points $(0,-1)$, $(0,1)$ and $(0,0)$

d.

- Show that $\forall k \in \mathbb{R} \setminus \{0, \ln 2\}, C_f^k = \left\{\left(x, \frac{x}{e^k - 1}\right) \in \mathbb{R}^2 : x \neq 0\right\}$
- Show that $C_f^0 = \{(0,y) \in \mathbb{R}^2 : y \neq 0\}$ and geometrically represent it
- Show that $C_f^1 = \left\{\left(x, \frac{x}{e-1}\right) \in \mathbb{R}^2 : x \neq 0\right\}$ and geometrically represent it
- Show that $C_f^2 = \left\{\left(x, \frac{x}{e^2-1}\right) \in \mathbb{R}^2 : x \neq 0\right\}$ and geometrically represent it

e.

- Show that $D_{g \circ f} = D_f$
- Show that $\forall (x,y) \in D_f, (g \circ f)(x,y) = \left(\frac{x}{y}, x - y\right)$
- Conclude that the characterization of $g \circ f$ is $\begin{array}{l} g \circ f : D_f \subset \mathbb{R}^2 \to \mathbb{R}^2 \\ (g \circ f)(x,y) = \left(\frac{x}{y}, x - y\right) \end{array}$

f.

- Show that, for example, $\nexists (x,y) \in D_f : (g \circ f)(x,y) = (1,0)$
- Conclude that $(1,0) \notin R_{g \circ f}$, which implies that $R_{g \circ f} \neq \mathbb{R}^2$

3.

a.

- Show, using the definition of limit of a function, that $\lim_{(x,y)\to(0,0)} f_1(x,y) = 0$
- Use in the proof, for example, the fact that, for every $\delta > 0$, all arguments of f which distance less than $\sqrt{\delta}$ from $(0,0)$ have images which distance less than δ from 0
- Show, by substitution or using the definition of limit of a function, that $\lim_{(x,y)\to(0,0)} f_2(x,y) = \frac{1}{4}$
- Conclude that $\lim_{(x,y)\to(0,0)} f(x,y) = \left(0,\frac{1}{4}\right) \in \mathbb{R}^2$, which implies that f is continuously extendable to $(0,0)$
- State that the characterization of h is $$h: \mathbb{R}^2 \to \mathbb{R}^2$$ $$h(x,y) = \begin{cases} f(x,y) & \text{if} \quad (x,y) \neq (0,0) \\ \left(0,\frac{1}{4}\right) & \text{if} \quad (x,y) = (0,0) \end{cases}$$

b.

- Show, using derivation rules or the definition of partial derivative, that $g'_y(1,1^-) = \frac{1}{2}$
- Show, using the definition of partial derivative, that $g'_y(1,1^+) = 0$
- Conclude that, as $g'_y(1,1^-) \neq g'_y(1,1^+)$, $\nexists g'_y(1,1)$, which implies that g is not differentiable at $(1,1)$

c.

- State that $(a, a^2 + b)$ is in the interior of the set of points at which g is defined by $f_2(x,y)$
- State that $f_2(x,y)$ does not depend of y
- Explain why the directional derivative along $(0,v)$ implies changing only y, which implies that $g'_{(0,v)}(a, a^2 + b) = 0$

d.

- State that $A = \left\{\left(\frac{1}{n},\frac{2}{n}\right) \in \mathbb{R}^2 : n \in \mathbb{N}\right\} \cup \{(0,0)\}$
- State that, as $B = \left\{\left(\frac{1}{n},\frac{2}{n}\right) \in \mathbb{R}^2 : n \in \mathbb{N}\right\}$ is a set of isolated points of A and $(0,0)$ is a boundary point of A, A is closed
- State that, as, for example, $B_3(0,0) \supset A$, A is closed
- Conclude that A is compact
- State that, as i is differentiable at $(0,0)$, it is continuous at $(0,0)$
- State that, as B is a set of isolated points, i is continuous in B
- Conclude that i is continuous in A
- State that, as A is compact and i is continuous in A, Weierstrass' theorem allows to conclude that i has a global minimum and a global maximum in A

4.

a. State that the proposition is true because $\lim u_n = 0^+$, which implies that $\forall b \in \mathbb{R}^+, \exists m \in \mathbb{N}: (n \geq m \Rightarrow u_n < b)$

b. State that the proposition is false because, for example, if $f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ is the function defined by $f(x) = \frac{1}{x}$, then $f^{-1} = f$

c. State that the proposition is true because, if (a,b,c) is in the graph of f, then there is no point (a,b,d), with $d \neq c$ which also is in it, and all open balls centered at (a,b,c) have at least one point of that type

d. State that the proposition is false because, for example, f is not continuous at 3, as any open ball centered at 3 has at least one point whose image distances 1 from $f(3)$

e. State that the proposition is false because, for example, if $f: \mathbb{R} \to \mathbb{R}^2$ and $g: \mathbb{R}^2 \to \mathbb{R}$ are the functions defined, respectively, by $f(x) = (x, 0)$ and $g(a,b) = a + b$, then $J_{g \circ f}(x) = [1]$ and $J_{f \circ g}(a,b) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

f. State that the proposition is false because, for example, $f: \mathbb{R}^2 \to \mathbb{R}$, defined by $\frac{x^3}{x^2+y^2}$ if $(x,y) \neq (0{,}0)$ and by 0 if $(x,y) = (0{,}0)$, is not differentiable at $(0{,}0)$, but $f'_x(0{,}0) = f'_y(0{,}0) = 0$, which implies that $A \cup B \neq \Omega$