1.5 A Few Aspects of Logic

Implications

In order to keep track of each step in a chain of logical reasoning, it often helps to use implication arrows.

Suppose P and Q are two propositions such that whenever P is true, then Q is necessarily true. In this case, we usually write

$$P \implies Q \qquad [*]$$

This is read as "P implies Q," or "if P, then Q," or "Q is a consequence of P." The symbol $\implies$ is an **implication arrow,** and it points in the direction of the logical implication. Here are some examples of correct implications.

Example 1.7

(a) $x > 2 \implies x^2 > 4$.

(b) $xy = 0 \implies x = 0$ or $y = 0$.

(c) x is a square $\implies$ x is a rectangle.

(d) x is a healthy person $\implies$ x is breathing.

Notice that the word "or" in mathematics means the "inclusive or," signifying that "P or Q" means "either P or Q or both."

All the propositions in Example 1.7 are open propositions, just as are most propositions encountered in mathematics. An implication $P \implies Q$ means that for each value of some variable for which P is true, Q is also true.

In certain cases where the implication [*] is valid, it may also be possible to draw a logical conclusion in the other direction:

$$Q \implies P$$

In such cases, we can write both implications together in a single **logical equivalence**:

$$P \iff Q$$

We then say that "P is equivalent to Q," or "P if and only if Q," or just "P iff Q." Note that the statement "P only if Q" expresses the implication $P \implies Q$, whereas "P if Q" expresses the implication $Q \implies P$.

Necessary and Sufficient Conditions

There are other commonly used ways of expressing that proposition P implies proposition Q, or that P is equivalent to Q. Thus, if proposition P implies proposition Q, we state that P is a "sufficient condition" for Q. After all, for Q to be true, it is sufficient that P is true. Accordingly, we know that if P is satisfied, then it is certain that Q is also satisfied. In this case, we say that Q is a "necessary condition" for P. For Q must necessarily be true if P is true. Hence,

> P is a **sufficient condition** for Q means: $P \Longrightarrow Q$
>
> Q is a **necessary condition** for P means: $P \Longrightarrow Q$

For example, if we formulate the implication in Example 1.7(c) in this way, it would read:

A necessary condition for x to be a square is that x be a rectangle.

or

A sufficient condition for x to be a rectangle is that x be a square.

The corresponding verbal expression for $P \iff Q$ is simply: *P is a necessary and sufficient condition for Q, or P if and only if Q, or P iff Q*. It is evident from this that it is very important to distinguish between the propositions "P is a necessary condition for Q" (meaning $Q \Longrightarrow P$) and "P is a sufficient condition for Q" (meaning $P \Longrightarrow Q$). To emphasize the point, consider two propositions:

1. Breathing is a necessary condition for a person to be healthy.
2. Breathing is a sufficient condition for a person to be healthy.

Evidently proposition 1 is true. But proposition 2 is false, because sick (living) people are still breathing. In the following pages, we shall repeatedly refer to necessary and sufficient conditions. Understanding them and the difference between them is a necessary condition for understanding much economic analysis. It is not a sufficient condition, alas!

Problems

1. Implications and equivalences can be expressed in ways that differ from those already mentioned. Use the implication or equivalence arrows to mark in which direction you believe the logical conclusions proceed in the following propositions:
   a. The equation $2x - 4 = 2$ is fulfilled only when $x = 3$.
   b. If $x = 3$, then $2x - 4 = 2$.
   c. The equation $x^2 - 2x + 1 = 0$ is satisfied if $x = 1$.
   d. If $x^2 > 4$, then $x > 2$ or $x < -2$, and conversely.
2. Consider the following six implications and decide in each case: (i) if the implication is true, and (ii) if the converse implication is true. (x, y, and z are real numbers.)
   a. $x = 2$ and $y = 5 \Longrightarrow x + y = 7$
   b. $(x-1)(x-2)(x-3) = 0 \Longrightarrow x = 1$
   c. $x^2 + y^2 = 0 \Longrightarrow x = 0$ or $y = 0$
   d. $x = 0$ and $y = 0 \Longrightarrow x^2 + y^2 = 0$
   e. $xy = xz \Longrightarrow y = z$
   f. $x > y^2 \Longrightarrow x > 0$
3. Consider the proposition $2x + 5 \geq 13$.
   a. Is the condition $x \geq 0$ necessary, sufficient, or both necessary and sufficient for the proposition to be satisfied?
   b. Answer the same question when $x \geq 0$ is replaced by $x \geq 50$.
   c. Answer the same question when $x \geq 0$ is replaced by $x \geq 4$.