Advanced Microeconomics

Spring 2024
Final Exam

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1. You have a total of 120 minutes (2 hours) to solve the exam.
2. The use of calculators is not allowed.
3. If you need additional space to answer a question, you can use the back *of the same page*.

Read each question carefully. Good luck!

I (5 points)

Consider a pure exchange economy with two agents, Agent A and Agent B. The preferences and initial endowments of the agents are as follows:

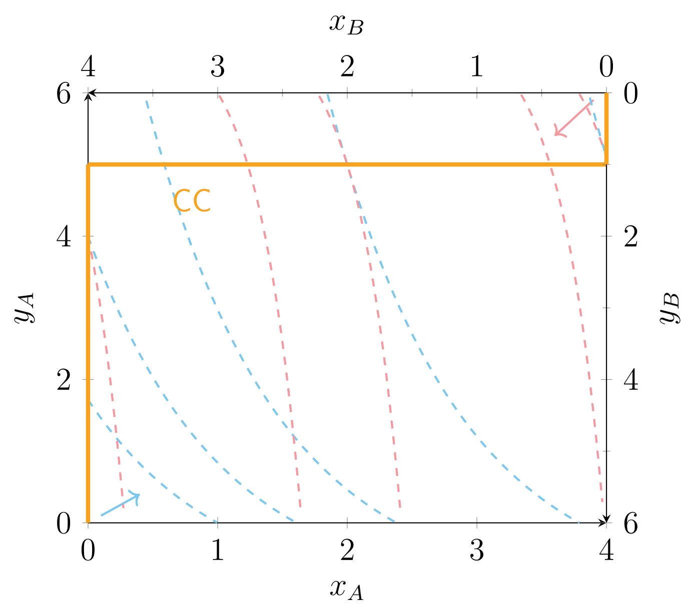
- Agent A has a utility function $u_A(x_A, y_A) = x_A + \ln(y_A + 1)$. Initially, Agent A has 2 unit of good x and 4 units of good y.
- Agent B has a utility function $u_B(x_B, y_B) = 3x_B + \ln(y_B + 1)$. Initially, Agent B has 2 units of good x and 2 units of good y.

a) **(2 points)** Graphically represent the contract curve. Explain your steps and reasoning clearly.

Solution: As preferences are strictly monotonic and strictly convex, we can apply the tangency condition:

$$
\begin{aligned}
MRS^A_{x,y} &= MRS^B_{x,y} \\
\frac{1}{1/(y_A+1)} &= \frac{3}{1/(y_B+1)} \\
y_A + 1 &= 3 \cdot (6 - y_A) + 3 \\
y_A &= 5
\end{aligned}
$$

It is immediately apparent that the function above contains neither of the origins on the edgeworth box (0,0) and (4, 6). A graphical analysis, however, shows that both origins are efficient points, as are also the points between them and the function above. As such, our contract curve will have the following representation:



Grading Criteria:

- 0.75 points for applying the tangency condition
- 0.75 points for the analytical expression of interior efficient points
- 0.5 points for recognizing the remaining efficient points

b) **(2 points)** Find the Walrasian equilibrium.

Solution:

Agent A's Demand:

$$\begin{cases} MRS^A_{x,y} = \frac{P_x}{P_y} \\ P_x.x_A + P_y.y_A = 2P_x + 4P_y \end{cases} \Leftrightarrow \begin{cases} y_A + 1 = \frac{P_x}{P_y} \\ P_x.x_A + P_y.y_A = 2P_x + 4P_y \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y_A = \frac{P_x}{P_y} - 1 \\ P_x.x_A + P_y.\left(\frac{P_x}{P_y} - 1\right) = 2P_x + 4P_y \end{cases} \Leftrightarrow \begin{cases} x^*_A = 1 + \frac{5P_y}{P_x} \\ y^*_A = \frac{P_x}{P_y} - 1 \end{cases}$$

if $y^*_A < 0 \Leftrightarrow \frac{P_x}{P_y} < 1 \rightarrow y^*_A = 0$ and $x^*_A = \frac{2P_x+4P_y}{P_x}$

Agent B's Demand:

$$\begin{cases} MRS^B_{x,y} = \frac{P_x}{P_y} \\ P_x.x_B + P_y.y_B = 2P_x + 2P_y \end{cases} \Leftrightarrow \begin{cases} 3(y_B + 1) = \frac{P_x}{P_y} \\ P_x.x_B + P_y.y_B = 2P_x + 2P_y \end{cases} \Leftrightarrow$$

$$\begin{cases} y_B = \frac{P_x}{3P_y} - 1 \\ P_x.x_B + P_y\left(\frac{P_x}{3P_y} - 1\right) = 2P_x + 2P_y \end{cases} \Leftrightarrow \begin{cases} x^*_B = \frac{5}{3} + \frac{3P_y}{P_x} \\ y^*_B = \frac{P_x}{3P_y} - 1 \end{cases}$$

if $y^*_B < 0 \Leftrightarrow \frac{P_x}{P_y} < 3 \rightarrow y^*_B = 0$ and $x^*_B = \frac{2P_x+2P_y}{P_x}$

General Equilibrium:

$$y_A^* + y_B^* = 6 \Leftrightarrow \frac{P_x}{P_y} - 1 + \frac{P_x}{3P_y} - 1 = 6 \Leftrightarrow$$
$$\Leftrightarrow p + \frac{p}{3} = 8 \Leftrightarrow 3p + p = 24 \Leftrightarrow$$
$$\Leftrightarrow p = 6$$

$$\left(x_A^*, y_A^*, x_B^*, y_B^*, \frac{P_x}{P_y}\right) = \left(\frac{11}{6}, 5, \frac{13}{6}, 1, 6\right)$$

Grading Criteria:

- 0.75 points for agent A's Demand (-0.1 for not recognizing the case where $y_A^* < 0$)
- 0.75 points for agent B's Demand (-0.1 for not recognizing the case where $y_B^* < 0$)
- 0.5 points for the market clearing conditions

c) **(1 point)** Without additional calculations, discuss whether the equilibrium allocation is Pareto efficient and whether it lies in the core of the economy.

Solution: As agents have weakly monotonic preferences, we can apply the First Welfare Theorem and thus the equilibrium is efficient. Furthermore, as the initial endowment is affordable, the equilibrium must also lie in the mutual advantages set, and thus also in the intersection between the MAS and the contract curve, i.e. the core.

Grading Criteria:

- 0.375 for applying the First Welfare Theorem
- 0.375 for recognizing the equilibrium lies in the Mutual Advantages Set
- 0.25 for defining the core and concluding the equilibrium belongs to it

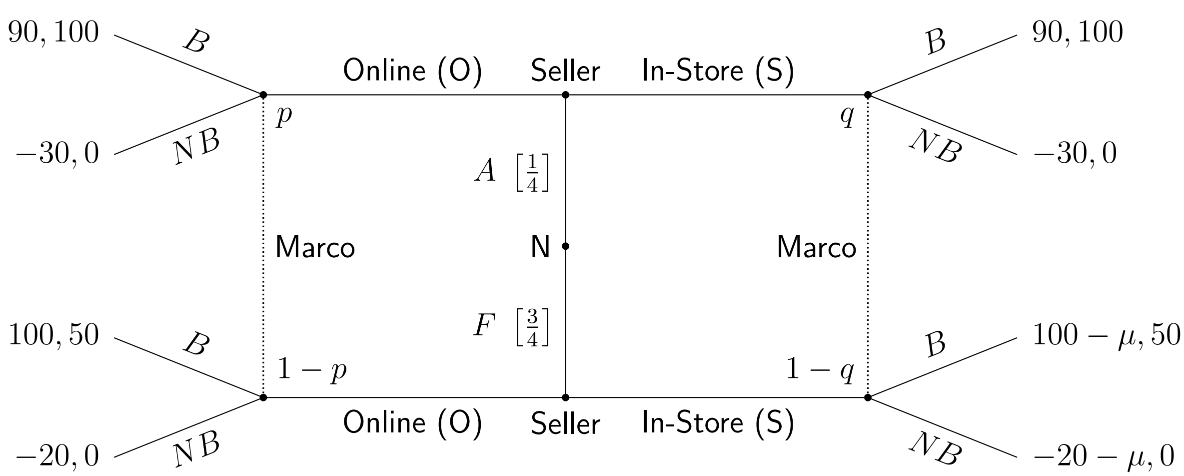
II (4.5 points)

Dr. Martins is a well-known company that sells boots. It is known that, in the boot market, $\frac{1}{4}$ of the boots are authentic (A) Dr. Martins and that the remaining $\frac{3}{4}$ are fake (F).
The boot seller may decide to sell them online (O) or in-store (S).

If the boots are sold in a store, sellers of authentic boots do not pay rent (as Dr. Martins offers them a retail space for free), while sellers of fake boots must pay a rent of μ. Moreover, the cost of producing a pair of authentic boots is 30, while the cost of producing a pair of fake boots is 20. There are no other costs besides the rent and the production costs. The selling price of any pair of boots is 120, regardless of quality and of how they are sold (online or in-store).
The payoff of a seller is simply the difference between revenues and costs. Revenue is 0 if no boots are sold; or the selling price of a pair of boots in case they are sold. The costs must be paid regardless of whether sale occurs.

Marco wants to buy a pair of Dr. Martins boots. After observing if the boots are sold online (O) or in-store (S), Marco decides whether to buy (B) or not to buy (NB) them. Marco puts a value of 220 on authentic Dr. Martins' boots and a value of 170 on the fake version. In case Marco buys the boots, his payoff is the difference between his valuation of each type of boots and the selling price. In case he does not buy the boots, his payoff is 0.

a) **(1.5 points)** Represent the game in extensive form.



Grading Criteria:

- 0.9 points for the structure
- 0.6 points for the payoffs

b) **(2 points)** For what values of μ is there a Perfect Bayesian equilibrium where sellers of authentic boots choose in-store shopping, and sellers of fake boots choose online shopping? In this case, can the choice of in-store shopping constitute an informative signal?
Solution:

Marco's Beliefs given (S,O):

$$p = P(A \mid O) = \frac{P(O \mid A) \times P(A)}{P(O)} = \frac{0 \times \frac{1}{4}}{\frac{3}{4}} = 0$$

$$q = P(A \mid S) = \frac{P(S \mid A) \times P(A)}{P(S)} = \frac{1 \times \frac{1}{4}}{\frac{1}{4}} = 1$$

Marco's Best Response given his beliefs:
After observing Online:

$$E_\pi(B) = 100 \times 0 + 50 \times 1 = \underline{50}$$

$$E_\pi(NB) = 0 \times 0 + 0 \times 1 = 0$$

After observing In-Store:

$$E_\pi(B) = 100 \times 1 + 50 \times 0 = \underline{100}$$

$$E_\pi(NB) = 0 \times 1 + 0 \times 0 = 0$$

Marcos's Best Response is (B,B)

Seller's Best Response given (B,B):
Authentic Seller:

$$\pi(O) = \underline{90} = \pi(S) = \underline{90}$$

Fake Seller:

$$\pi(O) = \underline{100} > \pi(S) = 100 - \mu$$

For any $\mu \geq 0$ the seller will have (S,O) as a best response. Thus, we have a separating PBE $[(S, O), (B, B), p = 0, q = 1]$, thus granting an informative signal.

Grading Criteria:

- 0.1 points for recognizing the strategy as (S,O)
- 0.2 points for each of Marco's beliefs
- 0.25 for Marco's best response after observing Online
- 0.25 for Marco's best response after observing In-Store
- 0.25 for the authentic seller's best response
- 0.25 for the fake seller's best response
- 0.25 for recognizing the seller will have (S,O) as a best response for any $\mu \geq 0$
- 0.25 for concluding there is a PBE and characterizing it

c) **(1 point)** Consider that $\mu = 20$. Is there a Perfect Bayesian equilibrium where both types of sellers choose in-store shopping?

Solution:

Marco's Beliefs given (S,S):

$$p = P(A \mid O) = \frac{P(O \mid A) \times P(A)}{P(O)} = \frac{0 \times \frac{1}{4}}{0} \rightarrow \text{Out of the equilibrium path, thus:}$$

$$p \in [0, 1]$$

$$q = P(A \mid S) = \frac{P(S \mid A) \times P(A)}{P(S)} = \frac{1 \times \frac{1}{4}}{1} = \frac{1}{4}$$

Marco's Best Response given his beliefs:

After observing Online:

$$E_\pi(B) = 100 \times p + 50 \times (1 - p) = \underline{50 + 50p}$$

$$E_\pi(NB) = 0 \times p + 0 \times (1 - p) = 0$$

After observing In-Store:

$$E_\pi(B) = 100 \times \frac{1}{4} + 50 \times \frac{3}{4} = \underline{62.5}$$

$$E_\pi(NB) = 0 \times \frac{1}{4} + 0 \times \frac{3}{4} = 0$$

Marcos's Best Response is (B,B)

Seller's Best Response given (B,B):

Authentic Seller:

$$\pi(O) = \underline{90} = \pi(S) = \underline{90}$$

Fake Seller:

$$\pi(O) = \underline{100} > \pi(S) = 100 - 20 = 80$$

The seller will have two best responses: (O, O) and (S, O). Thus, we do not have a pooling PBE where both types of sellers choose in-store shopping.

Grading Criteria:

- 0.05 points for recognizing the strategy as (S,S)
- 0.1 points for each of Marco's beliefs
- 0.125 for Marco's best response after observing Online
- 0.125 for Marco's best response after observing In-Store
- 0.125 for the authentic seller's best response
- 0.125 for the fake seller's best response
- 0.125 for recognizing the seller will not have (S,S) as a best response
- 0.125 for concluding there is no PBE

III (3.5 points)

Mercy is thinking about getting health insurance and her utility function is $U(w) = \sqrt{w}$. Mercy currently has a wealth of 36 and faces a possible loss of 20 if she becomes sick, which can happen with a probability of 50%. Let b be the number of units of insurance bought (where each unit pays 1 monetary unit in case of a loss).

a) **(0.5 points)** What is Mercy's attitude towards risk?

<u>**Solution:**</u> $U'(w) = \frac{1}{2\sqrt{w}}$; $U''(w) = -\frac{1}{4}w^{-\frac{3}{2}} < 0 \rightarrow$ Mercy is risk-averse.

Grading Criteria:

- 0.3 for the justification (for instance, showing that $U''(w) < 0$, $RP > 0$, $E(\pi) > CE$)
- 0.2 for the conclusion

b) **(1.5 points)** How much wealth would Mercy be willing to give up (with certainty) in order to eliminate the risk?

<u>**Solution:**</u>

$$E(\pi) = 0.5 \times 36 + 0.5 \times 16 = 26$$

$$0.5 \cdot U(36) + 0.5 \cdot U(16) = U(CE) \Leftrightarrow$$

$$\Leftrightarrow 0.5 \cdot 6 + 0.5 \cdot 4 = \sqrt{CE} \Leftrightarrow CE = 25$$

$$RP = E(\pi) - CE = 1$$

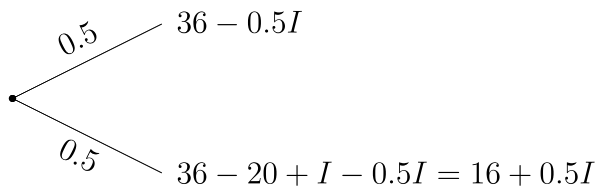
Mercy is willing to give up 1 unit of expected wealth with certainty in order to avoid the lottery's risk.

Grading Criteria:

- 0.75 for the CE
- 0.25 for the expected value
- 0.25 for the risk premium
- 0.25 for the conclusion

c) **(1.5 points)** If the insurance company is offering an actuarially fair insurance, **compute** how much insurance Mercy is going to buy.

<u>**Solution:**</u> With Actuarially Fair Insurance: $P = 0.5I$. Thus, the lottery Mercy faces is:



Mercy will solve:

$$\max_I E(U) = 0.5 \cdot \sqrt{36 - 0.5I} + 0.5 \cdot \sqrt{16 + 0.5I}$$

$$E(U)' = 0 \Leftrightarrow$$

$$\Leftrightarrow 0.5 \times \left(\frac{1}{2} \cdot (-0.5) \cdot (36 - 0.5I)^{-\frac{1}{2}}\right) + 0.5 \times \left(\frac{1}{2} \cdot (0.5) \cdot (16 + 0.5I)^{-\frac{1}{2}}\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow 36 - 0.5I = 16 + 0.5I \Leftrightarrow I = 20$$

Mercy will insure for 20, the full value of the loss.

Grading Criteria:

- 0.75 for the expected utility expression
- 0.5 for the derivative
- 0.25 for the conclusion

IV (3.5 points)

Clarice just hired Anne to work in her bookstore. Anne can spend her time slacking off (e=0) or working hard (e=1). Anne's utility is given by $u(w,e) = \sqrt{w} - e$. Assume that Anne has a reservation utility of 1. Moreover, Clarice only cares about the net profits of the bookstore. Gross profits can be 20 (thousand Euros) in case of a good quarter or 4 (thousand Euros) in case of a bad quarter. The probability of each outcome depends on Anne's choice of how to spend her time and is given by:

Profit level	$e = 1$	$e = 0$
$G = 20$	$3/4$	$1/2$
$B = 4$	$1/4$	$1/2$

a) **(1.5 points)** What effort level would Clarice want to implement if she could observe Anna's effort level?

Solution:
Low Effort Contract: $u(w_l, e = 0) \geq 1 \Leftrightarrow \sqrt{w_l} \geq 1 \Leftrightarrow w_l \geq 1$. Set $w_l = 1$ to minimize costs. $\rightarrow E(\pi) = 0.5 \cdot (20 - 1) + 0.5 \cdot (4 - 1) = 11$
High Effort Contract: $u(w_h, e = 1) \geq 1 \Leftrightarrow \sqrt{w_h} - 1 \geq 1 \Leftrightarrow w_h \geq 4$. Set $w_h = 4$ to minimize costs $\rightarrow E(\pi) = 0.75 \cdot (20 - 4) + 0.25 \cdot (4 - 4) = 12$
Clarice would implement a high level of effort, as it provides a greater level of expected profit.

Grading Criteria;

- 0.5 for low-effort case
- 0.5 for high-effort case
- 0.5 for expected profits and conclusion

b) **(2 points)** Assuming that Clarice cannot observe Anna's effort level, find the optimal contract that Clarice should offer.
Solution: Set wages based on outcomes (w_G, w_B)
To induce low-effort: Set a flat wage of $w = 1$ to minimize costs.
To induce high-effort:

$$\frac{3}{4} \cdot (\sqrt{w_G} - 1) + \frac{1}{4} \cdot (\sqrt{w_B} - 1) \geq \frac{1}{2} \cdot (\sqrt{w_G}) + \frac{1}{2} \cdot (\sqrt{w_B})$$

$$\frac{1}{4} \cdot \sqrt{w_G} - \frac{1}{4} \cdot \sqrt{w_B} - 1 \geq 0$$

Set $w_B = 0$ to minimize costs.

$$\frac{1}{4} \cdot \sqrt{w_G} - 1 \geq 0 \Leftrightarrow w_G \geq 16$$

Set $w_G = 16$ to minimize costs.

Inducing low-effort: $E(\pi) = \frac{1}{2}(20 - 1) + \frac{1}{2}(4 - 1) = 11$
Inducing high-effort: $E(\pi) = \frac{3}{4}(20 - 16) + \frac{1}{4}(4 - 0) = 4$

The optimal contract will be $(w_G, w_B) = (1, 1)$

Grading Criteria:

- 1 for high-effort wages
- 0.4 for low-effort wages
- 0.4 for expected profits;
- 0.2 for the conclusion.

V (3.5 points)

The 1939 Best Actress Oscar is up for sale in an auction. Fans of the movie value it at 1 million, but fans of the actress value it $\theta > 1$ (million). All of them have a reservation utility of 0 and one third of potential buyers are fans of the movie.

a) **(1 point)** Would an English auction be better than a Dutch auction (from the viewpoint of the seller)?

Solution: The english auction is strategically equivalent to the second-price action and the dutch auction is strategically equivalent to the first-price action.

There is revenue equivalence between first-price and second-price auctions if agents are risk neutral and valuations are independent. In such a case, the seller would be indifferent between the English Auction and the Dutch Auction.

In the event that agents are, rather, risk-averse, the first-price auction would have a greater expected revenue, thus being preferred to the second-price auction.

Grading Criteria:

- 0.25 for the strategic equivalence
- 0.5 for the revenue equivalence result
- 0.25 for the case of risk aversion

b) **(2.5 points)** Assuming that the buyers are risk-neutral and that their valuations are independent, find the values of θ for which the seller will never sell the Oscar to movie fans and describe the optimal auction mechanism in that case.

$$\max_{[M(\theta),M(\mu)]} E(\pi) = \frac{2}{3}M(\theta) + \frac{1}{3}M(\theta) \quad \text{s.t.} \quad \theta P(\theta) - M(\theta) \geq 0 \ (IR_\theta)$$

$$P(\mu) - M(\mu) \geq 0 \ (IR_\mu)$$

$$\theta P(\theta) - M(\theta) \geq \theta P(\mu) - M(\mu) \ (IC_\theta)$$

$$P(\mu) - M(\mu) \geq P(\theta) - M(\theta) \ (IC_\mu)$$

IC_θ and IR_μ guarantee IR_θ:

$$\theta P(\theta) - M(\theta) \geq \theta P(\mu) - M(\mu) \geq P(\mu) - M(\mu) \geq 0$$

$$\Rightarrow \theta P(\theta) - M(\theta) \geq 0$$

Moreover, IC_θ and IR_μ will both be binding and IC_μ can be ignored for now. The maximization problem is simplified to:

$$\max_{[M(\theta),M(\mu)]} E(\pi) = \frac{2}{3}M(\theta) + \frac{1}{3}M(\theta) \quad \text{s.t.} \quad P(\mu) - M(\mu) = 0 \ (IR_\mu)$$

$$\theta P(\theta) - M(\theta) = \theta P(\mu) - M(\mu) \ (IC_\theta)$$

$$\begin{cases} P(\mu) - M(\mu) = 0 \\ \theta P(\theta) - M(\theta) = \theta P(\mu) - M(\mu) \end{cases} \Leftrightarrow \begin{cases} M(\mu) = P(\mu) \\ M(\theta) = \theta P(\theta) + (1-\theta) P(\mu) \end{cases}$$

$$\frac{2}{3}(\theta P(\theta) + (1-\theta)P(\mu)) + \frac{1}{3}P(\mu)$$

$$\frac{1}{3}\left[2\theta P(\theta) + (3-2\theta)P(\mu)\right]$$

At this point, one may either recognize that for the item to never be sold to type μ, the coefficient of $P(\mu)$, $3-2\theta$, must be negative, or continue developing the expression:

$$\frac{1}{3}\left[2\theta\left(\frac{2}{3}P(\theta,\theta) + \frac{1}{3}P(\theta,\mu)\right) + (3-2\theta)\left(\frac{2}{3}P(\mu,\theta) + \frac{1}{3}P(\mu,\mu)\right)\right]$$

$$\frac{1}{9}\left[4\theta P(\theta,\theta) + 2\theta P(\theta,\mu) + (6-4\theta)P(\mu,\theta) + (3-2\theta)P(\mu,\mu)\right]$$

$$\frac{1}{9}\left[4\theta P(\theta,\theta) + 2\theta P(\theta,\mu) + (6-4\theta)(1-P(\theta,\mu)) + (3-2\theta)P(\mu,\mu)\right]$$

$$\frac{1}{9}\left[4\theta P(\theta,\theta) + (6\theta-6)P(\theta,\mu) + (3-2\theta)P(\mu,\mu) + 6 - 4\theta\right]$$

As $\theta > 1$, it is given that $6\theta - 6 > 0$ and thus $P(\theta,\mu) = 1$ and $P(\mu,\theta) = 0$. Not selling to μ, then, implies:

$$3 - 2\theta < 0 \Leftrightarrow \theta > 1.5$$

In such a case, the optimal mechanism will be:

$$\begin{cases} P(\theta,\theta) = 0.5 \\ P(\theta,\mu) = 1 \\ P(\mu,\theta) = 0 \\ P(\mu,\mu) = 0 \end{cases} \rightarrow \begin{cases} M(\theta) = \frac{2\theta}{3} \\ M(\mu) = 0 \end{cases}$$

Grading Criteria:

- 1 for the formulation
- 0.5 for the identification of binding constraints (including 0.25 for the proof)
- 1 for the solution (including 0.25 for complete description of mechanism and 0.25 for the range of theta)